THE

ELEMENTS

OF

ANALYTICAL MECHANICS.

SOLIDS AND FLUIDS.

BY

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PREFACE.

The plan of this edition is the same as the former one. It is designed especially for students who are beginning the study of Analytical Mechanics, and is preparatory to the higher works upon the same subject, and to Analytical Physics and Astronomy. The Calculus is freely used. I have sought to present the subject in such a manner as to familiarize the student with analytical processes. For this reason the solutions of problems have been treated as applications of general formulas. The solution by this method is often more lengthy than by special methods; still, it has advantages over the latter, because it establishes a uniformity in the process.

My experience has shown the importance of applying the fundamental equations to a great variety of problems. I have, therefore, in Article 24, and Chapters IV. and X., given a large number and a considerable variety of problems to be solved by the general equations under which they respectively fall.

In the revision I have been aided not only by my own experience with the use of the former edition in the class-room, but also by the friendly advice and criticism of several professors of colleges who have used the work. The result has been that several pages have been rewritten, some definitions changed, and the typographical errors corrected. Several new pages in the latter part of the work have been added. I am especially indebted to Professor E. T. Quimby, of Dartmouth College, Hanover, N. H., for his valuable suggestions and for assistance in reading the final proofs.

The *nature* of force remains as much a mystery as it was

when its principles were first recognized. Of its essential nature we shall probably remain forever in ignorance. We can only deal with the *laws* of its action. These laws are determined by observing the effects produced by a force. Force is the cause of an action in the physical world. The *results* of the action may be numerous and varied. Thus, force may produce pressure, tension, cohesion, adhesion, motion, affinity, polarity, electricity, etc. Or, to speak more properly, since force may be transmuted from one state to another, we would say that the above terms are names for the different manifestations of force.

The question "what is the correct measure of force" has taken different phases at different times. During the last century it was contended by some that momentum (Mv) was the correct measure, while others contended that it should be the work which it can do in a unit of time ($\frac{1}{2}Mv^2$). But as one has happily expressed it, "theirs was only a war of words;" for the real measure of force enters only as a factor in the expressions. Thus, if F be a constant force, the value of the momentum is Ft, see page 51, and of the work Fs, see page 45. At the present day some contend that the only measure of force is the motion which it produces, or would produce, in a unit of time. This is called the ABSOLUTE MEASURE, and THE ABSOLUTE UNIT OF FORCE *is the velocity which the force produces, or would produce, in a unit of mass in a unit of time if it acted during the unit with the intensity which it had at the instant considered.* If the intensity of the force were constant, the velocity which it produced at the end of the unit of time would be the required velocity. Hence, the absolute measure of any force acting on any mass is *the product of the mass into the acceleration;* and is the second member of equation (21). This is a correct measure, and is accepted as such by all writers on mechanics.

But those who contend that this is the only measure, necessarily deny that *weight*, or more generally *pounds*, is a measure. I contend that *pounds* is *a* measure of the intensity of a

force both statically and dynamically. Many authors maintain the same position. Indeed, it is probable that the position which I have taken can be *deduced* from any standard work on mechanics; but in some it is left to inference. Thus, in Smith's *Mechanics*, page 1, we find this terse and correct definition, "The *intensity* of a force may be measured, statically, by the pressure it will produce; dynamically, by the quantity of motion it will produce." I say this is correct, but I will add that the intensity of a force which produces a given motion is also measured by a pressure, or by something equivalent to a pressure, or to a pull. To those who will look at it analytically, it is only necessary to say that the first member of equation (21) is measured in *pounds*. If we know the absolute measure, we may easily find its value in *pounds*.

The *pound* here referred to is the result of the action of gravity upon a certain quantity of matter. The amount of matter having been fixed, either by a legal enactment or by common consent, and declared to be one pound at a certain place, its weight, as determined by a standard spring-balance at any other place, becomes a measure of the force of gravity as compared with the fixed place. This standard spring-balance may measure the intensity in pounds of any other force, whether the body upon which the force acts be at rest or in motion. If a perfectly free body were placed in a hollow space at the centre of the earth, at which place it would be devoid of weight, and pulled or pushed by a constant force, whose intensity, measured by a standard spring-balance, equaled the weight of an equal body on the surface of the earth, then would its motion be the same as that of a falling body. See page 24, Problem 7. In the forces of nature producing motion, there being no visible connection between the point of action of the force and the body upon which it acts, we are unable to *weigh* their intensity except by calculation. If the absolute measure is known, the *pounds* of intensity may be computed. The absolute measure of the force of gravity on a mass m is mg, and the weight of the body being W, we have $W = mg$. The sun acts upon the earth with a force which may be expressed by the absolute

measure, and also by a certain number of pounds of force. More than half of the examples in Article 24 involve an equality between *pounds of intensity* and the absolute measure of the force. The fact is, that, in case of motion, these quantities are co-relative. Since, then, it is correct to use the term *pound* as the measure of the intensity of a force whether the body be at rest or in motion, and since it is in common use, and the student is familiar with it, I prefer to consider a force as measured by a certain number of pounds. See Article 9. It is more simple, containing as it does only one element, than the absolute measure, which contains three elements—mass, velocity, and time.

There is another advantage in thus measuring force. Students frequently, and in some cases writers, use the expressions, "quantity of force," "amount of force," "force of a blow," etc., when they mean (or should mean) momentum, or work, or vis viva. In such cases an attempt to answer the question "how many pounds of force" would show at once that the quantity referred to was not *force*.

So much ambiguity, or at least indefiniteness, has arisen in regard to the term force, that I have rejected the terms "Impulsive Force" and "Instantaneous Force," and used the term "Impulse" instead of them. We know nothing of an instantaneous force, that is, one which requires no time for its action. I also reject the expression *force of inertia*. I do not believe that *inertia* is a *force*. To the question "The inertia of a body is how many pounds of force" there is no answer.

The term *moment of inertia* has no physical representation. The nearest approach to it is in the expression for the vis viva of a rotating body. In such problems the moment of inertia forms an important factor. The energy of a rotating body having a constant angular velocity is directly proportional to its moment of inertia in reference to its axis of rotation. See page 202. But *motion* is not necessary for its existence. See page 165. The expression appears in the discussion of numerous

statical problems, such as the flexure of a beam, the centre of pressure of a fluid, the centre of gravity of certain solids, etc. It is not the moment of a moment, although it may be so construed as to appear to be of that *form*. Some other term might be more appropriate. Even the expression *moment of the mass* would be less objectionable.

The subjects of *Centrifugal Force* and *Unbalanced Force* have been discussed of late in *Engineering*. Some assert that there is no such thing as a centrifugal force. Much unprofitable discussion may be avoided by strictly defining the terms used. If it is defined to be a force equal and opposite to the deflecting force, it will, at least, have an ideal existence, just as the resultant in statical problems has an ideal existence. But the vital question is, is the centrifugal force active when the deflecting force acts? Or, in other words, do both act upon the body at the same time? It seems, however, quite evident that if both acted upon the body at the same time they would neutralize each other, and the body would move in a straight line. Hence, in the movement of the planets, or of any free rotating body, there is no centrifugal force. But in the case of a locomotive running around a curve there may be both centripetal and centrifugal forces; the former acting against the locomotive to force it away from a tangent to the track; the latter, against the track, tending to force it outward. Wherever the force is conceived to act, whether just between the rail and wheel or at some other point, it is evident that both do not act upon the same body.

Similarly in regard to the *unbalanced force*. It is a convenient term to use, but, in a strict sense, an unbalanced force does not exist; for action and reaction are equal and opposite. But in reference to a particular body, all other conditions being ignored, the force may be unbalanced. Thus, when a ball is fired from a cannon, the force of the powder, considered in the direction of the motion of the ball only, is unbalanced; but the powder exerts an equal force in the opposite direction, and in that sense also is unbalanced. But when the entire effect of the

force in all directions is considered, the algebraic resultant is zero. In other words, the centre of gravity of the system, for forces acting between its integrant parts, remains constant.

These are some of the fundamental questions which will arise in the mind of the student as he studies the subject. Fortunately, it is not necessary for him to settle them beyond the question of a doubt before he proceeds with the subject. On some of these points scholars, who have made the subject a specialty, differ; and it is only after a careful consideration of the points involved that one can take an intelligent position in regard to them.

DeVolson Wood.

Hoboken, *August*, 1877.

GREEK ALPHABET.

Letters.	Names.	Letters.	Names.
Α α	Alpha	*Ν ν*	Nu
Β β	Bēta	Ξ ξ	Xi
Γ γ	Gamma	*Ο ο*	Omicron
Δ δ	Delta	*Π π*	Pi
Ε ε	Epsilon	*Ρ ρ*	Rho
Ζ ζ	Zēta	*Σ σ ς*	Sigma
Η η	Eta	*Τ τ*	Tau
Θ ϑ θ	Thēta	*Υ υ*	Upsilon
Ι ι	Iōta	Φ φ	Phi
Κ κ	Kappa	Χ χ	Chi
Λ λ	Lambda	Ψ ψ	Psi
Μ μ	Mu	*Ω ω*	Omega

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ANALYTICAL MECHANICS.

CHAPTER I.

DEFINITIONS, AND PRINCIPLES OF ACTION OF A SINGLE FORCE, AND OF FORCES ACTING ALONG THE SAME LINE.

1. MECHANICS treats of the laws of forces, and the equilibrium and motion of bodies under the action of forces. It has two grand divisions, Dynamics and Statics.

2. DYNAMICS treats of the motion of material bodies, and the laws of the forces which govern their motion.

3. STATICS treats of the conditions of the equilibrium of bodies under the action of forces.

There are many subdivisions of the subject, such as Hydrodynamics, Hydrostatics, Pneumatics, Thermodynamics, Molecular Mechanics, etc. That part of mechanics which treats of the relative motion of bodies which are so connected that one drives the other, such as wheels, pulleys, links, etc., in machinery, is called Cinematics. The motion in this case is independent of the intensity of the force which produces the motion.

Theoretic Mechanics treats of the effect of forces applied to material points or particles regarded as without weight or magnitude. Somatology is the application of theoretic mechanics to bodies of definite form and magnitude.

4. MATTER is that which receives and transmits force. In a physical sense it possesses extension, divisibility, and impenetrability.

Matter is not confined to the gross materials which we see and handle, but includes those substances by which sound, heat, light, and electricity are transmitted.

It is unnecessary in this connection to consider those refined speculations by which it is sought to determine the essential nature of matter. According to some of these speculations, *matter* does not exist, but is only a conception.

According to this view, bodies are forces, within the limit of which the attractive exceed the repulsive ones, and at the limits of which they are equal to each other.

But observation, long continued, teaches practically that matter is inert, that it has no power within itself to change its condition in regard to rest or motion; that when in motion it cannot change its rate of motion, nor be brought to rest without an external cause, and this cause we call FORCE. One also learns from observation that matter will transmit a force, as for instance a pull applied at one end of a bar or rope is transmitted to the other end; also a moving body carries the *effect* of a force from one place to another.

5. A BODY is a definite portion of matter. A particle is an infinitesimal portion of a body, and is treated geometrically as a point. A molecule is composed of several particles. An atom is an *indivisible* particle.

6. FORCE is that which tends to change the state of a body in regard to rest or motion. It moves or tends to move a body, or change its rate of motion.

We know nothing of the *essential* nature of force. We deal only with the *laws* of its action. These laws are deduced by observations upon the effects of forces, and on the hypothesis that *action and reaction are equal and opposite;* or, in other words, that the effect equals the cause. In this way we find that forces have different intensities, and that a relation may be established between them. It is necessary, therefore, to establish a UNIT. This may be assumed as the effect of any known force, or a multiple part thereof. The effect of all known forces is to produce a pull, or push, or their equivalents, and may be measured by pounds, or by something equivalent. The force of gravity causes the weight of bodies, and this is measured by pounds. We therefore assume that a STANDARD POUND is *the* UNIT of force.

The standard pound is established by a legal enactment, and has been so fixed that a cubic foot of distilled water at the level of the sea, at latitude 45 degrees, at a temperature of 62 degrees Fahrenheit, with the barometer at 30 inches, will weigh about 62.4 pounds avoirdupois.

The English standard pound was originally 5,760 troy grains. The grain was the weight of a certain piece of brass which was deposited with the clerk of the House of Commons. This was destroyed at the time of the burning of the House of Commons in 1834, after which it was decided that the legal

pound should be the weight of a certain piece of platinum, weighing 7,000 grains. This is known as the avoirdupois pound, and the troy pound ceased to be the legal standard, although both have remained in common use.

The legal standard pound in the United States is a copy of the English troy pound, and was deposited in the United States Mint in Philadelphia, in 1827, where it has remained. The avoirdupois pound, or 7,000 grains, is used in nearly all commercial transactions. The troy pound is a standard at 62 degrees Fahrenheit and 30 inches of the barometer.

The weight of a cubic inch of water at its maximum density, as accepted by the Bureau of Weights and Measures of the United States, is stated by Mr Hasler, in a report to the Secretary of the Treasury, 1842, to be 252.7455 grains. Mr. Hasler determined the temperature at which water has a maximum density, at 39.83 degrees Fahrenheit, but Playfair and Joule determined it to be 39.101° F.

The *exact* determination of the equivalent values of the units is very difficult, and has been the subject of much scientific investigation.—(See *The Metric System*, by F. A. P. Barnard, LL.D., New York, 1872.)

When a quantity can be measured directly, the *unit* is generally of the same quality as the thing to be measured: thus, the unit of time is time, as a day or second; the *unit* of length is length, as one inch, foot, yard, or metre; the *unit* of volume is volume, as one cubic foot; the *unit* of money is money; of weight is weight; of momentum is momentum; of work is work, etc.

When dissimilar quantities are used to measure each other a proportion must be established between them. It is commonly said that "the arc measures the angle at the centre," but it does not do it directly, since there is no ratio between them. The arc is a linear quantity, as feet or yards, or a number of times the radius, while the angle is the divergence of two lines, and is usually expressed in degrees. But angles are *proportional* to their subtended arcs; hence we have an equality of ratios, or

$$\frac{angle}{unit\ angle} = \frac{subtended\ arc}{arc\ which\ subtends\ the\ unit\ angle};$$

and since a semi-circumference, or π, subtends an angle of 180°, it is easy from the above equality of ratios to determine any angle when the arc is known, or *vice versâ*.

Similarly, the intensity of heat is not measured directly, but by its effect in expanding liquids or metals.

The magnetic force is measured by its effect upon a magnetic needle.

The intensities of lights by the relative shadows produced by them.

Similarly with forces, we measure them by their effects.

Dissimilar quantities, between which no proportion exists, do not measure each other. Thus feet do not measure time, nor money weight.

Pounds for commercial purposes represents quantities of matter; but when applied to forces it represents their intensities. In a strict sense, *pounds* does not measure *directly* the quantity of matter, but is always a measure of a force.

7. The line of action of a force is the line along which the force moves or tends to move a particle. If the particle is

acted upon by a single force, the line of action is straight This is also called the *action-line* of the force.

8. THE POINT OF APPLICATION of a force is the point at which it acts. This may be considered as at any point of its action-line. Thus, if a pull be applied at one end of a cord, the effect at the other end is the same as if applied at any intermediate point.

9. A FORCE is said to be given when the following elements are known:—

1st. Its magnitude (*pounds*);

2d. The direction of the line along which it acts (*action-line*);

3d. The direction along the action-line (+ or —); and,

4th. Its point of application.

A force may be definitely represented by a straight line; thus, its magnitude may be represented by the length $A\ B$, Fig. 1; its position by the position of the line $A\ B$; its direction along the line by the arrow-head at B, which indicates that the force acts from A towards B; and its point of application by the end A.

FIG. 1.

10. SPACE is indefinite extension, finite portions of which may be measured.

11. TIME is duration, and may be measured.

Probably no definition will give a better idea of the abstract quantities of *time* and *space* than that which is formed from experience.

12. A BODY is in motion when it occupies successive portions of space in successive instants of time. In all other cases it is at *absolute* rest. Motion in reference to another moving body is *relative*.

But a body may be at rest in reference to surrounding objects and yet be in motion. Thus, many objects on the surface of the earth, such as rocks, trees, etc., may be at rest in reference to objects around them, while they move with the earth through space. Observation teaches that there is probably no body at absolute rest in the universe.

13. MOTION IS UNIFORM when the body passes over equal portions of space in equal successive portions of time.

14. VARIABLE MOTION is that in which the body passes over unequal portions of space in equal times.

15. VELOCITY *is the rate of motion.* When the motion is *uniform* it is measured by the linear distance over which a body would pass in a unit of time; and when it is *variable* it is the distance over which it would pass if it moved with the rate which it had at the instant considered. The path of a moving particle is the line which it generates.

For uniform velocity, we have

$$v = \frac{s}{t}, \tag{1}$$

in which

$s =$ the space passed over;
$t =$ the time occupied in moving over the space s; and
$v =$ the velocity.

For variable velocity, we have

$$v = \frac{ds}{dt}. \tag{2}$$

EXAMPLES.

1. If a particle moves uniformly thirty feet in three seconds, what is its velocity?

2. If $s = at$, what is the velocity?

3. If $s = at^2 + bt$, what is the velocity at the time t, or at the end of the space s?

Here

$$v = \frac{ds}{dt} = 2\,at + b,$$

which is the answer to the first part. Find t from the given equation, and substitute in the expression for v, and it gives the answer to the second part; or

$$v = \sqrt{b^2 + 4\,as}.$$

4. If $s = 4t^3$, required the velocity at the end of five seconds.

5. If $3s^3 = 5t^2$, required the velocity at the end of ten seconds.

6. If $s = \frac{1}{2}gt^2$, what is the velocity in terms of the time and space?

7. If $at = e^{bs} - 1$, required the velocity in terms of the time and space.

16. ANGULAR VELOCITY *is the rate of angular movement.* If a particle moves around a point having either a constant or a variable velocity along its path, *the angular velocity is measured by the arc at a unit's distance which subtends the angle swept over in a unit of time by that radius vector which passes through the particle.*

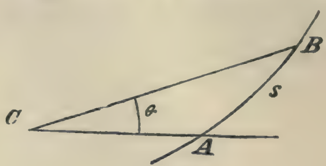

FIG. 2.

If $s = AB =$ the length of the path described;
$v =$ the velocity along the path AB;
$t =$ the time of the movement;
$r = CB =$ the radius vector;
$\theta =$ the circular arc at a unit's distance which subtends the angle ACB swept over by the radius vector in the time t; and
$\omega =$ the angular velocity;

Then, if the angular motion is *uniform,*

$$\omega = \frac{\theta}{t}. \tag{3}$$

If it is *variable,* then

$$\omega = \frac{d\theta}{dt}. \tag{4}$$

We also have,

$$ds = vdt = \sqrt{r^2 d\theta^2 + dr^2};$$

$$\therefore d\theta^2 = \frac{ds^2 - dr^2}{r^2} = \frac{v^2dt^2 - dr^2}{r^2}; \text{ and}$$

$$\omega = \frac{d\theta}{dt} = \frac{\sqrt{v^2 - \left(\frac{dr}{dt}\right)^2}}{r}. \tag{5}$$

17. ACCELERATION *is the rate of increase or decrease of the velocity.* It is a *velocity-increment.* The *velocity-increment* of an increasing velocity is considered positive, and that of a decreasing velocity, negative.

The measure of the acceleration, when it is uniform, is the amount by which the velocity is increased (or decreased) in a

unit of time. If the acceleration is variable, it is the amount by which the velocity would be increased in a unit of time, provided the rate of increase continued the same that it was at the instant considered.

Hence, if

$f =$ the measure of the acceleration (or briefly, the acceleration);

then, when the acceleration is uniform,

$$f = \frac{v}{t} = \frac{\frac{ds}{dt}}{t},$$

and hence, when it is variable,

$$f = \frac{dv}{dt} = \frac{d\frac{ds}{dt}}{dt} = \frac{d^2s}{dt^2}. \qquad (6)$$

We also have

$$f = \frac{d^2s}{dt^2} \times \frac{ds^2}{ds^2} = \frac{d^2s}{ds^2} \times \frac{ds^2}{dt^2} = v^2\frac{d^2s}{ds^2}. \qquad (6')$$

We thus see that the relation between space, time, and velocity are independent of the cause which produces the velocity.

Applications of Equation (6).

1. Suppose that the acceleration is constant.

Then in (6) f will be constant, and dt being the equicrescent variable, we have

$$f\int dt = \frac{1}{dt}\int d^2s,$$

$$\text{or } ft = \frac{ds}{dt} + C_1. \qquad (7)$$

But for $t = 0$, $\frac{ds}{dt} = v_0 =$ the initial velocity $\therefore C_1 = -v_0$; and (7) becomes

$$ds = ftdt + v_0dt.$$

Integrating again gives

$$s = \tfrac{1}{2}ft^2 + v_0t + C_2.$$

$$\text{But } s = s_0 \text{ for } t = 0 \ \therefore\ C_2 = s_0;$$

hence the final equation is

$$s = \tfrac{1}{2}ft^2 + v_0t + s_0; \tag{8}$$

which gives the relation between the space and time.

Again, multiplying both members of equation (6) by ds, we have

$$\frac{1}{dt^2}\int ds\, d^2s = f\int ds;$$

or

$$\frac{ds^2}{dt^2} = 2fs + v_0^2; \tag{9}$$

$$\therefore\ dt = \frac{ds}{\sqrt{v_0^2 + 2fs}};$$

and integrating, gives

$$t = \frac{\sqrt{v_0^2 + 2fs}}{f} + C. \tag{10}$$

Equation (7) gives the relation between the velocity and time, and equation (9) between the velocity and space.

If v_0 and s_0 are both zero, the preceding equations become

$$v = ft = \sqrt{2fs} = \frac{2s}{t}. \tag{11}$$

$$s = \tfrac{1}{2}ft^2 = \frac{v^2}{2f} = \tfrac{1}{2}vt. \tag{12}$$

$$t = \frac{v}{f} = \sqrt{\frac{2s}{f}} = \frac{2s}{v}. \tag{13}$$

We shall find hereafter that these formulas are applicable to all cases in which the *force* is constant and uniform.

2. Find the relation between the space and time when the acceleration is naught.

We have

$$\frac{d^2s}{dt^2} = 0.$$

Multiply by dt, integrate twice, and we have

$$s = s_0 + v_0t;$$

in which s_0 and v_0 are initial values; that is, the body will have

passed over a space s_0 before t is computed. v_0 is not only the initial but the constant uniform velocity. If $s_0 = 0$, then $s = v_0 t$.

3. If the acceleration varies directly as the time from a state of rest, required the velocity and space at the end of the time t.

Here $f = at$.

4. Determine the velocity when the acceleration varies inversely as the distance from the origin and is negative; or $f = -\frac{a}{s}$.

5. Determine the relation between the space and time when the acceleration is negative and varies directly as the distance from the origin; or $f = -bs$.

Equation (6) becomes

$$\frac{d^2s}{dt^2} = -bs.$$

Multiplying both members by ds, we have

$$\frac{ds\,d^2s}{dt^2} = -bs\,ds.$$

Integrating gives

$$v^2 = \frac{ds^2}{dt^2} = -bs^2 + C_1.$$

But $v = 0$ for $s = s_0$ $\therefore C_1 = bs_0^2$; and

$$\frac{ds^2}{dt^2} = b(s_0^2 - s^2) = v^2, \tag{14}$$

$$\text{or } b^{\frac{1}{2}}\,dt = \frac{ds}{(s_0^2 - s^2)^{\frac{1}{2}}}.$$

Integrating again gives

$$b^{\frac{1}{2}}\,t = \sin^{-1}\frac{s}{s_0} + C_2.$$

But $t = 0$ for $s = s_0$ $\therefore C_2 = -\frac{1}{2}\pi$;

$$\therefore s = s_0 \sin\left(tb^{\frac{1}{2}} + \tfrac{1}{2}\pi\right). \tag{15}$$

If $s = s_0$, $t = 0$, $2\pi b^{-\frac{1}{2}}$, $4\pi b^{-\frac{1}{2}}$.
" $s = 0$, $t = \frac{1}{2}\pi b^{-\frac{1}{2}}$, $\frac{3}{2}\pi b^{-\frac{1}{2}}$, $\frac{5}{2}\pi b^{-\frac{1}{2}}$, $\frac{7}{2}\pi b^{-\frac{1}{2}}$ $\frac{9}{2}\pi b^{-\frac{1}{2}}$, etc.;
" $s = - s_0$, $t =$ $\pi b^{-\frac{1}{2}}$, $3\pi b^{-\frac{1}{2}}$, $5\pi b^{-\frac{1}{2}}$.

This is an example of periodic motion, of which we shall have examples hereafter.

6. Determine the space when the acceleration diminishes as the square of the velocity.

When the acceleration is constant, the relation between the time, space, and velocity may be shown by a triangle, as in Fig. 3. Let AB represent the time, say four seconds. Divide it into four equal spaces, and each space will represent a second. Draw horizontal lines through the points of division and limit them by the inclined line AC. The horizontal lines will represent the corresponding velocities. Thus $v_2 = ge$ is the velocity at the end of the time t_2. The triangle Abc represents the space passed over during the first second, and ABC the space passed over during four seconds. The lines de, fh, and iC represent the accelerations for each second, which in this case are equal to each other, and equal to bc, which is the velocity at the end of the first second. Hence, *when the acceleration is uniform, the velocity at the end of the first second equals the acceleration.* This is also shown by Eq. (11); for if $t = 1$, $v = f$. Equations (12) and (13) may be deduced directly from the figure.

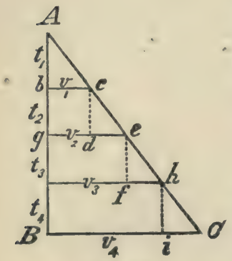

FIG. 3.

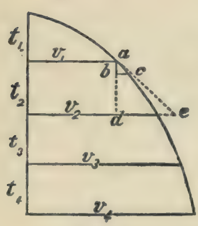

FIG. 4.

If acceleration constantly varies, the case may be represented as in Fig. 4. To find the acceleration at the end of the first second, draw a tangent ae to the curve at the point a, and drop the perpendicular ad, then will de be the acceleration. But $\frac{de}{ad} = \frac{bc}{ab} = \frac{dv}{dt} = f =$ the velocity-increment, which is the same as Equation (6).

18. RESOLVED VELOCITIES AND ACCELERATIONS. When the motion is along a known path and at a known rate, the projections of the velocities and accelerations upon other paths which are inclined to the given one will equal the product of these quantities by the cosine of the angle between the paths; that is,

$$v' = v\cos\theta = \frac{ds}{dt}\cos\theta, \text{ and } f' = \frac{d^2s}{dt^2}\cos\theta,$$

where v' and f' are on the new path, and θ the angle between the paths.

EXAMPLES.

1. If the velocity v is constant and along the line AB, which makes an angle θ with the line AC, then will the velocity projected on AC also be constant, and equal to $v\cos\theta$; and on the line BC, equal to $v\sin\theta$.

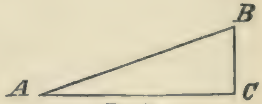

FIG. 5.

2. Let ABC be a parabola whose equation is $y^2 = 2px$. If a body describes the arc BC with such a varying velocity that its projection on BD, a tangent at B, is constant, required the velocity and the acceleration parallel to BE.

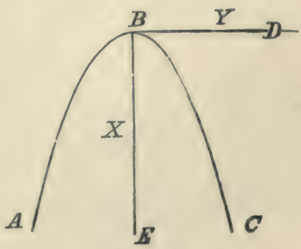

FIG. 6.

From the equation of the curve we have

$$\frac{dy}{dx} = \frac{p}{y}.$$

From the conditions of the problem we have

$$\frac{dy}{dt} = \text{constant} = v';$$

but
$$\frac{dx}{dt} = \frac{dy}{dt}\,\frac{dx}{dy} = v'\,\frac{y}{p} = v'\sqrt{\frac{2x}{p}},$$

which is the velocity parallel to x;

$$\therefore \frac{d^2x}{dt^2} = \frac{v'}{p}\frac{dy}{dt} = \frac{v'^2}{p};$$

hence the acceleration parallel to the axis of x will be constant.

Let ds be an element of the arc, then will the velocity along the arc be

$$v = \frac{ds}{dt} = \left(\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right)^{\frac{1}{2}} = v'\left(1 + \frac{2x}{p}\right)^{\frac{1}{2}}.$$

3. Determine the accelerations parallel to the co-ordinate axes x and y, so that a particle may describe the *arc* of a parabola with a constant velocity.

Let the equation of the parabola be

$$y^2 = 2px;$$

$$\therefore \frac{dy}{dx} = \frac{p}{y}.$$

The conditions of the problem give

$$\frac{ds}{dt} = \text{constant} = v.$$

But, $$\frac{ds}{dt} = \frac{dy}{dt}\cdot\frac{ds}{dy} = \frac{dy}{dt}\frac{\sqrt{dx^2 + dy^2}}{dy} = \frac{dy}{dt}\sqrt{1 + \frac{dx^2}{dy^2}}$$

$$= \frac{dy}{dt}\sqrt{1 + \frac{y^2}{p^2}} = v;$$

$$\therefore \frac{dy}{dt} = \frac{pv}{\sqrt{p^2 + y^2}};$$

and differentiating, gives

$$\frac{d^2y}{dt^2} = -\frac{pvy}{(p^2 + y^2)^{\frac{3}{2}}}\frac{dy}{dt}$$

$$= -\frac{p^2v^2y}{(p^2 + y^2)^2};$$

which being negative shows that the acceleration perpendicular to the axis of the parabola constantly diminishes.

Similarly we find

$$\frac{d^2x}{dt^2} = \frac{p^3v^2}{(y^2 + p^2)^2}.$$

4. A wheel rolls along a straight line with a uniform velocity; compare the velocity of any point in the circumference with that of the centre.

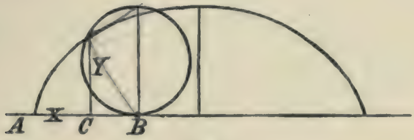

FIG. 7.

Let $v =$ the velocity of any point in the circumference,
$v' =$ the uniform velocity of the centre,
$r =$ the radius of the circle,
$x =$ the abscissa which coincides with the line on which it rolls, and
$y =$ the ordinate to any point of the cycloid.

Take the origin at A. The centre of the circle moves at the same rate as the successive points of contact B. The centre is vertically over B. The abscissa of the point of contact corresponding to any ordinate y of the cycloid, is $r\,versin^{-1}\frac{y}{r}$;

$$\therefore v' = \frac{d}{dt}\left(r\,versin^{-1}\frac{y}{r}\right) = \frac{r}{\sqrt{2ry - y^2}} \cdot \frac{dy}{dt};$$

$$\therefore \frac{dy}{dt} = v'\,\frac{\sqrt{2ry - y^2}}{r}.$$

The equation of the cycloid is

$$x = r\,versin^{-1}\frac{y}{r} - (2ry - y^2)^{\frac{1}{2}};$$

and from the theory of curves

$$ds^2 = dx^2 + dy^2 \therefore \frac{ds}{dy} = \sqrt{1 + \frac{dx^2}{dy^2}};$$

or, $$\frac{ds}{dy} = \sqrt{\frac{2r}{2r - y}};$$

and, $$v = \frac{ds}{dt} = \frac{dy}{dt} \cdot \frac{ds}{dy} = \sqrt{\frac{2y}{r}} \cdot v'.$$

If
$$y = 0, \quad v = 0;$$
$$y = r, \quad v = \sqrt{2}v';$$
$$y = 2r, \; v = 2v';$$
$$y = \tfrac{1}{2}r, \; v = v'.$$

Hence, at the instant that any point of the wheel is in contact with the straight line, it has no velocity, and the velocity at the highest point is twice that of the centre.

The velocity at any point of the cycloid is the same as if the wheel revolved about the point of contact, and with the same angular velocity as that of the generating circle.

For, the length of the chord which corresponds to the ordinate y is $\sqrt{2ry}$, and hence, if

$$v : 2v' :: \sqrt{2ry} : 2r;$$

we have
$$v = \sqrt{\frac{2y}{r}}\, v', \text{ as before found.}$$

19. GRAVITATION is that natural force which mutually draws two bodies towards each other. It is supposed to extend to every particle throughout the universe according to fixed laws. The force of gravity above the surface of the earth diminishes as the square of the distance from the centre increases, but within the surface it varies directly as the distance from the centre. If a body were elevated one mile above the surface of the earth it would lose nearly $\frac{1}{2000}$ of its weight, which is so small a quantity that we may consider the force of gravity for small elevations above the surface of the earth as practically constant. But it is variable for different points on the surface, being least at the equator, and gradually increasing as the latitude increases, according to a law which is approximately expressed by the formula

$$g = 32.1726 - 0.08238 \cos 2L,$$

in which $L =$ the latitude of the place,

$g =$ the acceleration due to gravity at the latitude L, or simply the force of gravity, and

32.1726 ft. $=$ the force of gravity at latitude 45 degrees.

From this we find that

at the equator $g = g_0 = 32.09022$ feet, and
at the poles $g = g_{90} = 32.25498$ feet.

The varying force of gravity is determined by means of a pendulum, as will be shown hereafter. It is impossible to determine the exact law of relation between the force of gravity at different points on the surface of the earth, for it is not homogeneous nor an exact ellipsoid of revolution. The delicate observations made with the pendulum show that any assumed formula is subject to a small error. (See *Mécanique Céleste*, and *Puissant's Géodésie*.)

Substituting g for f in equations (11), (12), and (13), we have the following equations, which are applicable to bodies falling freely in *vacuo*:—

$$\left.\begin{aligned} v &= gt = \sqrt{2gs} = \frac{2s}{t}; \\ s &= \tfrac{1}{2}gt^2 = \frac{v^2}{2g} = \tfrac{1}{2}vt; \\ t &= \frac{v}{g} = \sqrt{\frac{2s}{g}} = \frac{2s}{v}. \end{aligned}\right\} \qquad (16)$$

EXAMPLES.

1. A body falls through a height of 200 feet; required the time of descent and the acquired velocity. Let $g = 32\frac{1}{6}$ feet.

Ans. $t = 3.53$ seconds.
$v = 113.31$ feet.

2. A body is projected upward with a velocity of 1000 feet per second; required the height of ascent when it is brought to rest by the force of gravity.

Ans. $s = 15{,}544$ feet, nearly.

3. A body is dropped into a well and four seconds afterwards it is heard to strike the bottom. Required the depth, the velocity of sound being 1130 feet per second.

Ans. 231 feet.

4. A body is projected upward with a velocity of 100 feet per second, and at the same instant another body is let fall from a height 400 feet above the other body; at what point will they meet?

5. With what velocity must a body be projected downward that it may in t seconds overtake another body which has already fallen through a feet?

$$Ans.\ v = \frac{a}{t} + \sqrt{2ag}.$$

6. Required the space passed over by a falling body during the n^{th} second.

20. Mass is quantity of matter. If we conceive that a quantity of matter, say a cubic foot of water, earth, stone, or other substance, is transported from place to place, without expansion or contraction, the quantity will remain the same, while its weight may constantly vary. If placed at the centre of the earth it will weigh nothing; if on the moon it will weigh less than on the earth, if on the sun it will weigh more; and if at any place in the universe its weight will be directly as the attractive force of gravity, and since the acceleration is also directly as the force of gravity, we have

$$\frac{W}{g} = \text{constant},$$

for the same mass at all places. This ratio for any contemporaneous values of W and g may be taken as the measure of the *mass*, as will be shown in the two following articles. The weight in these cases must be determined by a spring balance or its equivalent.

21. Dynamic measure of a force. Conceive that a body is perfectly free to move in the direction of the applied force, and that a constant *uniform* force, which acts either as a *pull* or *push*, is applied to the body. It will at the end of one second produce a certain velocity, which call $v_{(1)}$. If now forces of different intensities be applied to the same body they will produce velocities in the same time which are proportional to the forces; or

$$\mathrm{f} \propto v_{(1)},$$

in which f is the applied force.

Again, if the same forces are applied to bodies having different masses, producing the same velocities in one second, then will the forces vary directly as the masses, or,

$$\mathrm{f} \propto M.$$

Hence, generally, if *uniform*, *constant* forces are applied to different masses producing velocities $v_{(1)}$ in one second, then

$$\mathrm{f} \propto Mv_{(1)};$$

or, in the form of an equation,

$$\mathrm{f} = cMv_{(1)}; \qquad (17)$$

where c is a constant to be determined.

If the forces are *constantly varying*, the velocities generated at the end of one second will not measure the intensities at any instant, but according to the above reasoning, the *rate of variation of the velocity* will be one of the elements of the measure of the force. Hence if

F = a *variable* force;
M = the mass moved;
$\frac{dv}{dt} = f$ = the rate of variation of the velocity; or *velocity-increment;*

and, $\frac{dv}{dt}$ be substituted for $v_{(1)}$ in equation (17), reducing by equation (6), we have

$$F = cMf = cM\frac{dv}{dt} = cM\frac{d^2s}{dt^2}. \qquad (18)$$

From this we have

$$cM = \frac{F}{f};$$

hence the value of cM is expressed in terms of the constant ratio of the force F to that of the acceleration f.

FIG. 8.

To determine this ratio experimentally I suspended a weight, W, by a very long fine wire. The wire should be long, so that the body will move practically in a straight line for any arc through which it will be made to move, and it should be very small, so that it will contain but little mass. By means of suitable mechanism I caused a constant force, F, to be applied horizontally to the body, thus causing it to move sidewise, and determined

the space over which it passed during the first second. This equalled one-half the acceleration (see the first of equations (12) when $t = 1$). I found when $F = \frac{1}{20} W$, that $f = 1.6$ feet, nearly; and for $F = \frac{1}{10} W$, $f = 3.2$ feet, nearly; and similarly for other forces; hence

$$cM = \tfrac{1}{32} W, \text{ nearly.}$$

But the ratio of F to f is determined most accurately and conveniently by means of falling bodies; for $f = g =$ the acceleration due to the force of gravity, and W the weight of the body (which is a measure of the statical effect of the force of gravity upon the body), hence

$$cM = \frac{W}{g}; \tag{19}$$

in which the values of W and g must be determined at the same place; but that place may be anywhere in the universe. The value of c is assumed, or the relation between c and M fixed arbitrarily.

If $c = 1$, we have

$$M = \frac{W}{g}; \tag{20}$$

and this is the expression for the *mass*, which is nearly if not quite universally adopted. This in (18) gives

$$F = M\frac{d^2s}{dt^2} = \frac{W}{g}\frac{d^2s}{dt^2}; \tag{21}$$

and hence THE DYNAMIC MEASURE OF THE PRESSURE WHICH MOVES A BODY *is the product of the mass into the acceleration.* This is sometimes called an accelerating force.

If there are retarding forces, such as friction, resistance of the air or water, or forces pulling in the opposite direction; then the first member F, is the measure of the unbalanced forces in pounds, and the second member is its dynamic equivalent.

22. UNIT OF MASS. If it is assumed that $c = 1$, as in the preceding article, the *unit of mass* is virtually fixed. In (20) if $W = 1$ and $g = 1$, then $M = 1$; that is, *a unit of mass* is the quantity of matter which will weigh one pound at that

place in the universe where the acceleration due to gravity is one. If a quantity of matter weighs $32\frac{1}{6}$ lbs. at a place where $g = 32\frac{1}{6}$ feet, we have

$$M = \frac{32\frac{1}{6}}{32\frac{1}{6}} = 1;$$

hence on the surface of the earth a body which weighs $32\frac{1}{6}$ pounds (nearly) is *a unit of mass.*

It would be an *exact* unit if the acceleration were exactly $32\frac{1}{6}$ feet.

In order to illustrate this subject further, suppose that we make the *unit of mass* that of a *standard pound.* Then equation (19) becomes

$$c \,.\, 1 = \frac{1}{g_0},$$

in which g_0 is the value of g at the latitude of 45 degrees. This value resubstituted in the same equation gives

$$M = \frac{g_0}{g} W,$$

and these values in equation (18) give

$$F = \frac{1}{g_0} Mf = \frac{W}{g} \frac{d^2s}{dt^2};$$

the final value of which is the same as (21).

Again, if the *unit of mass* were the weight of one cubic foot of distilled water at the place where $g_0 = 32.1801$ feet, at which place we would have $W = 62.3791$, and (19) would give

$$c \,.\, 1 = \frac{62.3791}{32.1801},$$

and this in the same equation gives

$$M = \frac{32.1801}{62.3791} \cdot \frac{W}{g},$$

and these values in (18) give

$$F = \frac{W}{g} \frac{d^2s}{dt^2}, \text{ as before.}$$

23. DENSITY *is the mass of a unit of volume.*

If $M =$ the mass of a body;
$V =$ the volume; and
$D =$ the density;

then if the density is uniform, we have

$$D = \frac{M}{V}.$$

If the density is variable, let

$\delta =$ the density of any element, then

$$\delta = \frac{dM}{dV};$$

$$\therefore M = \int \delta \, dV \qquad (22)$$

from which the mass may be determined when δ is a known function of V.

EXAMPLES.

1. In a prismatic bar, if the density increases uniformly from one end to the other, being zero at one end and 5 at the other, required the total mass.

Let $l =$ the length of the bar;
$A =$ the area of the transverse section; and
$x =$ the distance from the zero end;

then will

$\frac{5}{l} =$ the density at a unit's distance from the zero end;

$\frac{5}{l}x =$ the density at a distance x; and

$dV = Adx$,

$$\therefore M = A\int_0^l \frac{5xdx}{l} = \frac{5}{2}Al.$$

2. In a circular disc of uniform thickness, if the density at a unit's distance from the centre is 2, and increases directly as the distance from the centre, required the mass when the radius is 10.

3. In the preceding problem suppose that the density increases as the square of the radius, required the mass.

4. In the preceding problem if the density is two pounds per cubic foot, required the weight of the disc.

5. If in a cone, the density diminishes as the cube of the distance from the apex, and is *one* at a distance one from the apex, required the mass of the cone.

Having established a unit of density, we might properly say that mass is a certain number of *densities*.

24. Applications of Equation (21).

[Obs.—If, for any cause, it is considered desirable to omit any of the matter of the following article, the author urges the student to at least establish the equations for the acceleration for each of the 31 examples here given. This part belongs purely to mechanics. The reduction of the equations belongs to mathematics. It would be a good exercise to establish the fundamental equations for all these examples, before making any reductions. Such a course serves to impress the student with the distinction between mechanical and mathematical *principles*.]

1*st. If a body whose weight is* 50 *pounds is moved horizontally by a constant force of* 10 *pounds, required the velocity acquired at the end of* 10 *seconds and the space passed over during that time, there being no friction nor other external resistance, and the body starting from rest.*

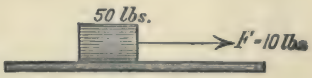

Fig. 9.

Here

$$M = \frac{W}{g} = \frac{50}{32\frac{1}{6}} \text{ lbs., and}$$

$$F = 10 \text{ lbs.};$$

hence (21) gives

$$\frac{d^2s}{dt^2} = \frac{F}{M} = \frac{193}{30}.$$

Multiply by dt and integrate, and

$$\frac{ds}{dt} = v = \frac{193}{30}\,t + (C_1 = 0).$$

The second integral is

$$s = \frac{193}{60} t^2 + (C_2 = 0);$$

and hence for $t = 10$ seconds, we have

$$v = 64.33 + \text{feet.}$$
$$s = 321.66 + \text{feet.}$$

2d. Suppose the data to be the same as in the preceding example, and also that the friction between the body and the plane is 5 *pounds. Required the space passed over in* 10 *seconds.*

Here $F = (10 - 5)$ pounds.

$$\therefore \frac{d^2s}{dt^2} = \frac{F}{M} = \frac{193}{60}.$$

3d. Suppose that a body whose weight is 50 *pounds is moved horizontally by a weight of* 10 *lbs., which is attached to an inextensible, but perfectly flexible string which passes over a wheel and is attached at the other end to the body. Required the distance passed over in* 10 *seconds, if the string is without weight, and no resistance is offered by the wheel, plane, or string.*

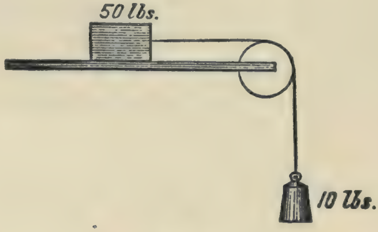

FIG. 10.

In this case gravity exerts a force of 10 pounds to move the mass, or $F = 10$ lbs., and the mass moved is that of both bodies, or $M = (50 + 10) \div 32\frac{1}{6}$.

$$\frac{d^2s}{dt^2} = \frac{F}{M} = \frac{193}{36}.$$

The integration is performed as before.

Ans. $s = 268.05$ feet.

4th. Find the tension of the string in the preceding example.

The tension will equal that *force* which, if applied directly to the body, as in Ex. 1, will produce the same acceleration as in the preceding example.

Let $P = 10$ pounds;

$W = 50$ pounds;

$T =$ tension;

$\frac{P + W}{g} =$ the mass in the former example; and

$\frac{W}{g} =$ the mass moved by the tension.

Hence, from Equation (21),

$$\frac{P + W}{g} f = P; \text{ and}$$

$$\frac{W}{g} f = T.$$

Eliminate f, and we find

$$T = \frac{WP}{W + P};$$

$$\therefore T = 8.33 \text{ lbs.}$$

What must be the value of P so that the tension will be a maximum or a minimum, $P + W$ being constant?

5th. *In example* 3, *what must be the weight of* P *so that the tension shall be* $(\frac{1}{n})^{\text{th}}$ *part of* P?

Ans. $P = (n-1)\ W.$

6th. *If a body whose weight is* W *falls freely in a vacuum by the force of gravity, determine the formulas for the motion.*

Here $Mg = W$ and the moving force $F = W$;

$$\therefore \frac{W}{g}\frac{d^2s}{dt^2} = W;$$

$$\text{or, } \frac{d^2s}{dt^2} = g.$$

The integrals of this equation will give Equations (16), when the initial space and velocity are zero. Let the student deduce them.

7th. Suppose that the moving pressure (pull or push) equals the weight of the body, required the velocity and space.

Here $Mg = W$ and $F = W$, hence the circumstances of motion will be the same as in the preceding example.

The forces of nature produce motion without *apparent* pressure, but this example shows that their effect is the same as that produced by a push or pull whose intensity equals the weight of the body, and hence both are measured by *pounds*, or their equivalent.

8th. If the force F is constant, show that $Ft = Mv$; also $Fs = \frac{1}{2}Mv^2$, and $\frac{1}{2}Ft^2 = Ms$. If F is variable we have $Mv = \int Fdt$.

9th. Suppose that a piston, devoid of friction, is driven by a constant steam-pressure through a portion of the length of a cylinder, at what point in the stroke must the pressure be instantly reversed so that the full stroke shall equal the length of the cylinder, the cylinder being horizontal?

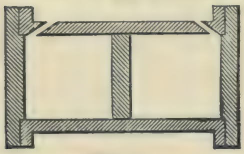

FIG. 11.

At the middle of the stroke. Whatever velocity is generated through one-half the stroke will be destroyed by the counter pressure during the other half.

10th. If the pressure upon the piston is 500 pounds, weight of the piston 50 pounds, and the friction of the piston in the cylinder 100 pounds, required the point in the stroke at which the pressure must be reversed that the stroke may be 12 inches.

The uniform effective pressure for driving the piston is $500 - 100 = 400$ lbs., and the uniform effective force for stopping the motion is $500 + 100 = 600$ pounds. The velocity generated equals the velocity destroyed, and the velocity destroyed equals that which would be generated in the same space by a force equal to the resisting force; hence if

$F =$ the effective moving force;
$s =$ the space through which it acts;
$v =$ the resultant velocity;
$F' =$ the resisting force; and
$s' =$ the space through which it acts;

then, from the expression in Example 8, we have

$$Fs = \tfrac{1}{2}Mv^2,$$

and $$F's' = \tfrac{1}{2}Mv^2,$$

$$\therefore Fs = F's',$$

or, $$F : F' :: s' : s.$$

In the example, $F = 400$ lbs., and $F' = 600$ lbs. Let $x =$ the distance from the starting point to the point where the pressure must be reversed. Then

$$600 : 400 :: x : 12 - x, \therefore x = 7\tfrac{1}{5} \text{ inches.}$$

11*th. If in the preceding example the piston moves vertically up and down, required the point at which the pressure must be instantly reversed so that the full stroke shall be* 12 *inches.*

The effective driving pressure upward will be $500 - 100 - 50 = 350$ pounds, and the retarding force will be $500 + 100 + 50 = 650$ pounds, and during the down-stroke the driving force is $500 + 50 - 100 = 450$ pounds, and the retarding force is $500 - 50 + 100 = 550$ pounds.

12*th. A string passes over a wheel and has a weight P attached at one end, and W at the other. If there are no resistances from the string or wheel, and the string is devoid of weight, required the resulting motion.*

FIG. 12.

Suppose $W > P$;

then

$$F = W - P, \text{ and}$$

$$M = \frac{W + P}{g};$$

$$\therefore \frac{d^2s}{dt^2} = \frac{F}{M} = \frac{W - P}{W + P}g.$$

By integrating, we find

$$v = \frac{W - P}{W + P}gt,$$

and, $$s = \tfrac{1}{2}\frac{W - P}{W + P}gt^2.$$

13*th. Required the tension of the string in the preceding example.*

The tension equals the weight P, plus the force which will produce the acceleration

$$\frac{W-P}{W+P}g$$

when applied to raise P vertically. The mass multiplied by the acceleration is this moving force, or

$$\frac{P}{g}\cdot\frac{W-P}{W+P}g;$$

hence the tension is

$$P+\frac{W-P}{W+P}P=\frac{2WP}{W+P}.$$

Similarly, it equals W minus the accelerating force, or

$$W-\frac{W-P}{W+P}W=\frac{2WP}{W+P}.$$

A complete solution of this class of problems involves the mass of the wheel and frictions, and will be considered hereafter.

14*th. A string passes over a wheel and has a weight* P *attached to one end and on the other side of the wheel is a weight* W, *which slides along the string. Required the friction between the weight* W *and the string, so that the weight* P *will remain at rest. Also required the acceleration of the weight* W.

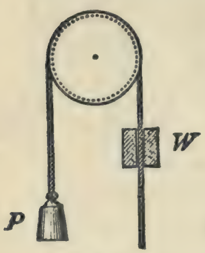

FIG. 13.

The friction $= P$;

$$Mg = W;$$

$$\text{and, } F = W - P;$$

$$\therefore \frac{d^2s}{dt^2} = \frac{F}{M} = \frac{W-P}{W}g;$$

hence,

$$v = \frac{W-P}{W}gt,$$

and,

$$s = \tfrac{1}{2}\frac{W-P}{W}gt^2.$$

15*th*. *In the preceding example, if W were an animal whose weight is less than P, required the acceleration with which it must ascend, so that P will remain at rest.*

16*th*. *If the weight W descend along a rough rope with a given acceleration, required the acceleration with which the body P must ascend or descend on the opposite rope, so that the rope may remain at rest, no allowance being made for friction on the wheel.*

(The ascent must be due to climbing up on the cord, or be produced by an equivalent result.)

17*th*. *A particle moves in a straight line under the action of a uniform acceleration, and describes spaces* s *and* s′ *in* t[th] *and* t′[th] *seconds respectively, determine the accelerating force and the velocity of projection.*

Let $v_0 =$ the velocity of projection, and
$f =$ the acceleration;

then

$$f = \frac{s' - s}{t' - t},$$

and

$$v_0 = \frac{s'(2t-1) - s(2t'-1)}{2(t - t')}.$$

If $\frac{s'}{s} = \frac{2t'-1}{2t\ -1}$, then $v_0 = 0$.

18*th*. *If a perfectly flexible and perfectly smooth rope is placed upon a pin, find in what time it will run itself off.*

If it is perfectly balanced on the pin it will not move, unless it receive an initial velocity. If it be unbalanced, the weight of the unbalanced part will set it in motion. Suppose that it is balanced and let

$v_0 =$ the initial velocity,
$2l =$ the length of the rope,
$w =$ the weight of a unit of length, and
$t =$ the time.

Take the origin of coördinates at the end of the rope at the instant that motion begins. When one end has descended s feet, the other has ascended the same amount, and hence the

unbalanced weight will be $2ws$. The mass moved will be $2wl \div g$; hence we have

$$\frac{d^2s}{dt^2} = \frac{F}{M} = \frac{2\ ws}{2\ wl}g = \frac{g}{l}s.$$

Multiply by ds and integrate, and we have

$$\frac{ds^2}{dt^2} = v^2 = \frac{g}{l}s^2 + (C = v_0^2);$$

$$\therefore \sqrt{\frac{g}{l}}dt = \frac{ds}{\sqrt{\frac{l}{g}v_0^2 + s^2}};$$

Integrating again, gives

$$t = \sqrt{\frac{l}{g}}\log\left\{\frac{s + \sqrt{\frac{l}{g}v_0^2 + s^2}}{\sqrt{\frac{l}{g}}v_0}\right\}$$

$$= \sqrt{\frac{l}{g}}\log\left\{\frac{l + \sqrt{\frac{l}{g}v_0^2 + l^2}}{\sqrt{\frac{l}{g}}v_0}\right\}, \text{ if } s = l.$$

19*th*. *If a particle moves towards a centre of force which* ATTRACTS *directly as the distance from the force, determine the motion.*

Let μ = the absolute force; that is, the acceleration at a unit's distance from the centre due to the force; and

s = the distance;

then

$$\frac{d^2s}{dt^2} = -\mu s.$$

A force is considered positive in whatever direction it acts, and the *plus* sign indicates that its *direction of action* is the same as that of the positive ordinate *from* the origin of coördinates, and the *negative* sign action in the opposite direction. If a be the initial value of s, we have (see Ex. 5, Art. 17):

$$v = \sqrt{\mu(a^2 - s^2)};$$

$$t = \mu^{-\frac{1}{2}}\sin^{-1}(\frac{s}{a} - \tfrac{1}{2}\pi);$$

and the velocity at the centre of the force is found by making $s = 0$, for which we have,

$$v = a\sqrt{\mu};$$

and $$t = -\frac{1}{2}\mu^{-\frac{1}{2}}\pi,\ \frac{1}{2}\mu^{-\frac{1}{2}}\pi,\ \frac{3}{2}\mu^{-\frac{1}{2}}\pi, \text{ etc.},$$

hence, the time is independent of the initial distance.

It may be proved that within a homogeneous sphere the attractive force varies directly as the distance from the centre. Hence, if the earth were such a sphere, and a body were permitted to pass freely through it, it would move with an accelerated velocity from the surface to the centre, at which point the velocity would be a maximum, and it would move on with a retarded velocity and be brought to rest at the surface on the opposite side. It would then return to its original position, and thus move to and fro, like the oscillations of a pendulum.

The acceleration due to gravity at the surface of the earth being g, and r being the radius, the absolute force is

$$\mu = \frac{g}{r};$$

$$\therefore v = r\sqrt{\mu} = \sqrt{g.r};$$

and the time of passing from surface to surface on the equator would be

$$t = \pi\sqrt{\frac{r}{g}} = 3.1416\sqrt{\frac{20,923,161}{32.09025}} = 42\text{m. } 1.6 \text{ sec.}$$

The *exact* dimensions of the earth are unknown. The semi-polar axis of the earth is, as determined by

Bessel	20,853,662 ft.
Airy	20,853,810 ft.
Clarke	20,853,429 ft.

The equatorial radius is not constant, on account of the elevations and depressions of the surface. There are some indications that the general form of the equator is an ellipse. Among the more recent determinations are those by Mr. Clarke, of England (1873), and his result given below is considered by him as the most probable *mean*. The equatorial radius, is according to

Bessel	20,923,596 ft.
Airy	20,923,713 ft.
Clarke	20,923,161 ft.

The determination of the force of gravity at any place is subject to small errors, and when it is *computed* for different places the result may differ from the actual value by a perceptible amount.

The force of gravity at any particular place is *assumed* to be constant, but all we can assert is that if it is variable the most delicate observations have failed to detect it. But it is well known that the surface of the earth is constantly undergoing changes, being elevated in some places and depressed in others, and hence, assuming the law of gravitation to be exact and universal, we cannot escape the conclusion that the force of gravity at every place on its surface changes, and although the change is exceedingly slight, and the total change may extend over long periods of time, it may yet be possible, with apparatus vastly more delicate than that now used, to measure this change. It seems no more improbable than the solution of many problems already attained—such for instance, as determining the relative velocities of the earth and stars by means of the spectroscope.

20th. Suppose that a coiled spring whose natural length is A B, is compressed to B C. If one end rests against an immovable body B, and the other against a body at C, which is perfectly free to move horizontally, what will be the time of movement from C to A, and what will be the velocity at A?

FIG. 14.

It is found by experiment that the resistance of a spring to compression varies directly as the amount of compression, hence the action of the spring in pushing the body, will, in reference to the point A, be the same as an attractive force which varies directly as the distance, and hence it is similar to the preceding example. But if the spring is not attached to the particle the motion will not be periodic, but when the particle has reached the point A it will leave the spring and proceed with a uniform velocity. If the spring were destitute of mass, it would extend to A, and become instantly at rest, but because of the mass in it, the end will pass A and afterwards recoil and have a periodic motion. If the body be *attached* to the spring, it will have a periodic motion, and the solution will be similar to the one in the Author's *Resistance of Materials*, Article 19.

Take the origin at A, s being counted to the left; suppose that 5 pounds will compress the spring one inch, and let the total compression be $a = 4$ inches. Let $W =$ the weight of the body $= 10$ pounds.

The force at the distance of one foot from the origin being 60 pounds, the force at s feet will be $60s$ pounds.

Hence, $$\frac{W}{g}\frac{d^2s}{dt^2} = -F = -60s;$$

or, $$\frac{d^2s}{dt^2} = -193s;$$

from which we find that

$$t = \frac{\frac{1}{2}\pi}{\sqrt{193}},$$

and $$v = 4.6 + \text{feet}.$$

21*st. Suppose that in the preceding problem a body whose weight is M' is at B, and another M'' at C, both being perfectly free to move horizontally, required the time of movement that the distance between them shall be equal to AB; and the resultant velocities of each.*

Take the origin at any convenient point, say in the line of the bodies and at a distance x' to the left of M', and let x'' be the abscissa of M''; b the length of the spring after being compressed an amount a, and μ the force in pounds which will compress it the first unit; then the tension of the spring when the length is s will be $\mu(a + b - s)$; hence we have

$$s = x'' - x',$$

$$M''\frac{d^2x''}{dt^2} = \mu(a + b - s),$$

$$M'\frac{d^2x'}{dt^2} = -\mu(a + b - s).$$

From the first,

$$\frac{d^2s}{dt^2} = \frac{d^2x''}{dt^2} - \frac{d^2x'}{dt^2};$$

substituting,

$$\frac{d^2s}{dt^2} = \mu\frac{M' + M''}{M'M''}(a + b - s);$$

integrating,

$$\frac{ds^2}{dt^2} = \mu\frac{M' + M''}{M'M''}(2as + 2bs - s^2) + C_1,$$

But $v = 0$ for $s = b$; $\therefore\ C_1 = -\mu \dfrac{(M' + M'')(2a + b)b}{M'M''}$;

and integrating again gives

$$\sqrt{\mu \frac{M' + M''}{M'M''}}\,.\,t = \text{vers}^{-1}\frac{s - b}{a} + (C_2 = 0);$$

and, making $s = a + b$, we have

$$t = \tfrac{1}{2}\pi\sqrt{\frac{1}{\mu}\cdot\frac{M'M''}{M' + M''}},$$

which, as in the preceding example, is independent of the amount of compression of the spring.

To find the relation between the absolute velocities,

Let s' = the space passed over by M', and
s'' = the space passed over by M'';

then since the moving force is the same for both, we have

$$M'\frac{d^2s'}{dt^2} = M''\frac{d^2s''}{dt^2}.$$

Integrating, gives

$$M'v' = M''v''.$$

22*d*. *Suppose that the force varies directly as the distance from the centre of force and is* REPULSIVE.

Then

$$\frac{d^2s}{dt^2} = \mu s;$$

$$\therefore s = \frac{v_0}{2\sqrt{\mu}}\left(e^{t\sqrt{\mu}} - e^{-t\sqrt{\mu}}\right),$$

in which v_0 is the initial velocity.

23*d*. *Suppose that the force varies inversely as the square of the distance from the centre of the force and is* ATTRACTIVE.

[This is the law of universal gravitation, and is known as the law of the *inverse squares*. While it is rigidly true, so far as we know, for every

particle of matter acting upon any other particle, it is not rigidly true for finite bodies acting upon other bodies at a finite distance, except for *homogeneous spheres*, or spheres composed of homogeneous shells. The earth being neither homogeneous nor a sphere, it will not be *exactly* true that it attracts external bodies with a force which varies inversely as the square of the distance from the centre, but the deviations from the law for bodies at great distances from the earth will not be perceptible. We assume that the law applies to all bodies above the surface of the earth, the centre of the force being at the centre of the earth.]

Let the problem be applied to the attraction of the earth, and

$r =$ the radius of the earth;
$g =$ the force of gravity at the surface;
$\mu =$ the absolute force; and
$s =$ the distance from the centre;

then

$$\mu = gr^2;$$

and

$$\frac{d^2s}{dt^2} = -\frac{\mu}{s^2}.$$

Multiply by ds and integrate; observing that for $s = a$, $v = 0$ and we have

$$\frac{ds^2}{dt^2} = 2\mu\left(\frac{1}{s} - \frac{1}{a}\right) \qquad (a)$$

$$= 2\mu\,\frac{as - s^2}{as^2};$$

$$\therefore \left(\frac{2\mu}{a}\right)^{\frac{1}{2}} dt = \frac{-\,sds}{(as - s^2)^{\frac{1}{2}}};$$

using the negative sign, because t and s are inverse functions of each other.

The second member may be put in a convenient form for integration by adding and subtracting $\frac{1}{2}a$ to the numerator and arranging the terms. This gives

$$\frac{\frac{1}{2}a - s - \frac{1}{2}a}{(as - s^2)^{\frac{1}{2}}}\,ds$$

$$= \frac{a - 2s}{2(as - s^2)^{\frac{1}{2}}}\,ds - \frac{ads}{2(as - s^2)^{\frac{1}{2}}};$$

the integral of which is

$$(as - s^2)^{\frac{1}{2}} - \tfrac{1}{2}a \text{ versin}^{-1} \frac{2s}{a} + C.$$

But when $\qquad s = a, t = 0 \therefore C = \tfrac{1}{2}a\pi;$

$$\therefore t = \left(\frac{a}{2\mu}\right)^{\frac{1}{2}} \left\{ \left(as - s^2 \right)^{\frac{1}{2}} + a \cos^{-1} \left(\frac{s}{a}\right)^{\frac{1}{2}} \right\} \qquad (b)$$

[From the circle we have $\pi - \text{versin}^{-1} \dfrac{2s}{a} = \pi - \cos^{-1}\left(1 - \dfrac{2s}{a}\right) =$

$$\cos^{-1}\left(-\left(1 - \frac{2s}{a}\right) \right) = \cos^{-1} \left(\frac{2s}{a} - 1\right).$$

From trigonometry we have $2 \cos^2 y - 1 = \cos 2y$.

Let $2y = \cos^{-1}\left(\dfrac{2s}{a} - 1\right)$, then

$$\cos 2y = \frac{2s}{a} - 1; \therefore \cos^2 y = \frac{s}{a}, \text{ and } y = \cos^{-1} \sqrt{\frac{s}{a}}; \text{ and}$$

$$2y = 2 \cos^{-1}\sqrt{\frac{s}{a}}; \text{ or } \pi - \text{versin}^{-1}\frac{2s}{a} \Big].$$

From (a) it appears that for $s = 0, v = \infty$; hence the velocity at the centre will be infinite when the body falls from a finite distance.

If $s = a = \infty, v = 0$. If a body falls freely from an infinite distance to the earth, we have in equation (a)

$$a = \infty; \text{ and}$$
$$s = r = \text{the radius of the earth};$$
$$\therefore v = \left(\frac{2\mu}{r}\right)^{\frac{1}{2}},$$

for the velocity at the surface. But $\dfrac{\mu}{r^2} = g$;

$$\therefore v = (2gr)^{\frac{1}{2}}.$$

If $g = 32\frac{1}{6}$ feet and $r = 3962$ miles, we have

$$v = \left(\frac{64\frac{1}{3} \times 3962}{5280}\right)^{\frac{1}{2}} = 6.95 \text{ miles}.$$

Hence the maximum velocity with which a body can reach the earth is less than seven miles per second.

24*th*. *Suppose that the force is* ATTRACTIVE *and varies inversely as the* n^{th} *power of the distance.*

Then

$$\frac{d^2s}{dt^2} = -\frac{\mu}{s^n};$$

$$\therefore \frac{ds^2}{dt^2} = \frac{2\mu}{n-1}\left(\frac{1}{s^{n-1}} - \frac{1}{a^{n-1}}\right);$$

and integrating, gives

$$t = \left(\frac{n-1}{2\mu}\right)^{\frac{1}{2}} a^{\frac{1}{2}(n-1)} \int_a^x (a^{n-1} - s^{n-1})^{-\frac{1}{2}} s^{\frac{1}{2}(n-1)} ds.$$

According to the *tests of integrability* this may be integrated when

$$n = \ldots \frac{5}{7}, \frac{3}{5}, \frac{1}{3}, -1, \frac{3}{1}, \text{ or } \frac{5}{3} \ldots \text{ etc.},$$

$$\text{or } n = \ldots \frac{3}{4}, \frac{2}{3}, \frac{1}{2}, 0, 2, \text{ or } \frac{3}{2} \ldots\ldots \text{ etc.}$$

25*th*. *Let the force vary inversely as the square root of the distance and be* ATTRACTIVE. (This is one of the special cases of the preceding example.)

We have

$$\frac{d^2s}{dt^2} = -\frac{\mu}{s^{\frac{1}{2}}};$$

$$\therefore \frac{ds^2}{dt^2} = 4\mu\,(a^{\frac{1}{2}} - s^{\frac{1}{2}});$$

$$\text{or, } 2\mu^{\frac{1}{2}}dt = \frac{-ds}{(a^{\frac{1}{2}} - s^{\frac{1}{2}})^{\frac{1}{2}}}.$$

The negative sign is taken because t and s are inverse functions of each other.

Add and subtract $\dfrac{2\sqrt{a}}{3\sqrt{s}\sqrt{a^{\frac{1}{2}}-s^{\frac{1}{2}}}}$ and we have

$$2\sqrt{\mu}\,dt = \left[\frac{-\sqrt{s}}{\sqrt{s}\sqrt{a^{\frac{1}{2}}-s^{\frac{1}{2}}}} + \frac{2\sqrt{a}}{3\sqrt{s}\sqrt{a^{\frac{1}{2}}-s^{\frac{1}{2}}}} - \frac{2\sqrt{a}}{3\sqrt{s}\sqrt{a^{\frac{1}{2}}-s^{\frac{1}{2}}}}\right]ds$$

$$= \left[\frac{-3\sqrt{s} + 2\sqrt{a}}{3\sqrt{s}\,\sqrt{a^{\frac{1}{2}} - s^{\frac{1}{2}}}} - \frac{2\sqrt{a}}{3\sqrt{s}\,\sqrt{a^{\frac{1}{2}} - s^{\frac{1}{2}}}} \right] ds$$

$$\therefore t = \frac{2}{3\sqrt{\mu}} \left[s^{\frac{1}{2}}(a^{\frac{1}{2}} - s^{\frac{1}{2}})^{\frac{1}{2}} + 2\sqrt{a}\,(a^{\frac{1}{2}} - s^{\frac{1}{2}})^{\frac{1}{2}} \right]$$

$$= \frac{2}{3\sqrt{\mu}} \left(s^{\frac{1}{2}} + 2a^{\frac{1}{2}} \right) \left(a^{\frac{1}{2}} - s^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

26th. Suppose that the force is ATTRACTIVE *and varies inversely as the distance.*

Hence

$$\frac{d^2s}{dt^2} = -\frac{\mu}{s};$$

$$\therefore \frac{ds^2}{dt^2} = 2\mu \log \frac{a}{s};$$

in which $s = a$ for $v = 0$. Hence the time from $s = a$ to $s = 0$, is

$$t = \frac{1}{\sqrt{2\mu}} \int_a^0 \frac{ds}{\left(\log \frac{a}{s} \right)^{\frac{1}{2}}} = a \left(\frac{\pi}{2\mu} \right)^{\frac{1}{2}}.$$

Let $\left(\log \frac{a}{s} \right)^{\frac{1}{2}} = y$; then for $s = a$, $y = 0$ and for $s = 0$, $y = \infty$. Squaring and passing to exponentials, we have

$$\log \frac{a}{s} = y^2 \quad \therefore \frac{a}{s} = e^{y^2}, \quad \text{or } s = a\,e^{-y^2};$$

$$\therefore \; ds = -\,a e^{-y^2}.\, 2y\, dy;$$

$$\therefore t = a \left(\frac{2}{\mu} \right)^{\frac{1}{2}} \int_0^\infty e^{-y^2}\, dy = a \left(\frac{\pi}{2\mu} \right)^{\frac{1}{2}}.$$

This is called a *gamma-function*, and a method of integrating it is as follows :—

Since functions of the same *form* integrated between the same limits are independent of the variables and have the same value, therefore

$$\int_0^\infty e^{-y^2} dy = \int_0^\infty e^{-t^2} dt;$$

$$\text{and} \int_0^\infty e^{-y^2} dy \int_0^\infty e^{-t^2} dt = \left[\int_0^\infty e^{-y^2} dy\right]^2.$$

Also the left hand member will be of the same value if the sign of integration be placed over the whole of it, since the actual integration will be performed in the same order; hence

$$\left[\int_0^\infty e^{-y^2} dy\right]^2 = \int_0^\infty \int_0^\infty e^{-y^2-t^2} dy\, dt$$

$$= \int_0^\infty \int_0^\infty te^{-t^2(1+u^2)}\, dt\, du;$$

in which $y = tu$; $\therefore\ dy = t\, du$. Integrating in reference to t, we have

$$\frac{-e^{-t^2(1+u^2)}}{2(1+u^2)} du,$$

which for $t = \infty$ becomes zero, and for $t = 0$ becomes $\dfrac{du}{2(1+u^2)}$, and the integral of this is $\frac{1}{2}\tan^{-1}u$, which is zero for $u = 0$, and $\frac{1}{4}\pi$ for $u = \infty$;

$$\therefore \int_0^\infty e^{-y^2} dy = \frac{1}{2}\sqrt{\pi}.$$

(See also *Méc. Céleste*, p. 151 [1534 O].)

Or we may proceed as follows :—

$$\text{Let } e^{-t^2} = x\ \therefore\ dt = -(-\log x)^{-\frac{1}{2}} \frac{dx}{2x};$$

$$\therefore \int_0^\infty e^{-t^2} dt = \int_1^0 -\tfrac{1}{2}(-\log x)^{-\frac{1}{2}} dx.$$

Let $x = a^{y^2}$ and consider a less than unity; then $\log a$ will be negative, and $\log x = y^2(-\log a)$;

$$\therefore\ dx = a^{y^2} 2y\, dy\, (-\log a);$$

which substituted above gives

$$\int_0^\infty e^{-t^2} dt = \int_1^0 -\tfrac{1}{2}(-\log x)^{-\frac{1}{2}} dx = (-\log a)^{\frac{1}{2}} \int_0^\infty a^{y^2} dy.$$

Dividing by $(-\log a)^{\frac{1}{2}}$ and multiplying both sides by $-\frac{1}{2}\,da$, we have

$$\int_1^0 -\tfrac{1}{2}(-\log a)^{-\frac{1}{2}} da \int_1^0 -\tfrac{1}{2}(-\log x)^{-\frac{1}{2}} dx = \int_0^\infty \int_1^0 -\tfrac{1}{2}\, a^{y^2} dy\, da.$$

Integrating the second member first in regard to a, gives

$$-\tfrac{1}{2}\frac{a^{y^2+1}}{y^2+1}\,dy;$$

which between the limits of 0 and 1 gives $\frac{1}{2}\dfrac{dy}{1+y^2}$; the integral of which is $\frac{1}{2}\tan^{-1} y$ which between the limits of ∞ and 0 gives $\frac{1}{4}\pi$.

$$\therefore \int_1^0 -\tfrac{1}{2}(-\log a)^{-\frac{1}{2}}\, da \int_1^0 -\tfrac{1}{2}(-\log x)^{-\frac{1}{2}} dx$$

$$= \left[\int_1^0 -\tfrac{1}{2}(-\log x)^{-\frac{1}{2}} dx\right]^2 = \tfrac{1}{4}\pi;$$

$$\therefore \int_0^\infty e^{-t^2} dt = \tfrac{1}{2}\sqrt{\pi}.$$

(See *Méc. Céleste*, Vol. iv. p. 487, Nos. [8319] to [8331]. Chauvenet's *Spherical Astronomy*, Vol. i. p. 152. Todhunter's *Integral Calculus*. Price's *Infinitesimal Calculus*.)

Sometimes the integration of an exponential quantity becomes apparent by first differentiating a similar one. Thus, to integrate $t\,e^{-t^2} dt$, first differentiate e^{-t^2}. We have $d\,e^{-t^2} = e^{-t^2} d(-t^2) = e^{-t^2}(-2\,tdt) = -2te^{-t^2} dt$;

$$\therefore \int de^{-t^2} = -2\int te^{-t^2} dt.$$

But the first member is the integral of the differential, and hence is the quantity itself, or e^{-t^2}, and hence the required integral is $-\frac{1}{2}\,e^{-t^2}$.

27*th. Suppose that two bodies have their centres at A and A′ respectively, and* ATTRACT *a particle at p with forces which vary as the distances from A and A′.*

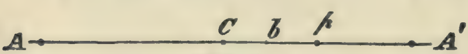

FIG. 15.

Let C be midway between A and A';

$Cp = c$;

$AC = CA' = a$;

$Cb = s =$ any variable distance;

and let $\mu = \mu' =$ the absolute forces of the bodies A and A respectively.

Then

$$\frac{d^2s}{dt^2} = \mu(a-s) - \mu(a+s) = -2\mu s;$$

$$\therefore \frac{ds^2}{dt^2} = 2\mu(c^2 - s^2);$$

and integrating again gives

$$s = c \cos t\sqrt{2\mu}.$$

28*th*. *Suppose that a particle is projected with a velocity u into a medium which resists as the square of the velocity; determine the circumstances of motion.*

Take the origin at the point of projection, and the axis s to coincide with the path of the body.

Let $\mu =$ the absolute resistance—or the resistance of the medium when the velocity is unity;

then $\mu\left(\frac{ds}{dt}\right)^2 =$ the resistance for any velocity;

$$\therefore \frac{d^2s}{dt^2} = -\mu\left(\frac{ds}{dt}\right)^2;$$

or,

$$\frac{d\left(\frac{ds}{dt}\right)}{\frac{ds}{dt}} = -\mu ds.$$

And integrating between the initial limits, $s = 0$ for $\frac{ds}{dt} = u$, and the general limits, we have

$$\log\frac{ds}{dt} - \log u = -\mu s;$$

or, $$\log \frac{\frac{ds}{dt}}{u} = -\mu s;$$

$$\therefore \frac{ds}{dt} = ue^{-\mu s};$$

or, $$udt = e^{\mu s} ds.$$

Integrating again, observing that $t = 0$ for $s = 0$, we have

$$\mu ut = e^{\mu s} - 1.$$

The velocity becomes zero only when $s = \infty$.

29*th*. *A heavy body falls in the air by the force of gravity, the resistance of the air varying as the square of the velocity; determine the motion.*

Take the origin at the starting point, and

Let $\kappa =$ the resistance of the body for a unit of velocity;
$s =$ the distance from the initial point, positive downwards;
$t =$ the time of falling through distance s;

then $\frac{ds}{dt} = 0$ for t and $s = 0$;

$\frac{ds}{dt} = v$ for $t = t$ and $s = s$; and

$\kappa\left(\frac{ds}{dt}\right)^2 =$ the resistance of the air at any point, and acts upwards;

and $g =$ the accelerating force downward;

hence, the resultant acceleration is the difference of the two, or

$$\frac{d^2s}{dt^2} = g - \kappa\left(\frac{ds}{dt}\right)^2; \qquad (a)$$

or, $$\frac{d\left(\frac{ds}{dt}\right)}{\kappa\, dt} = \frac{g}{\kappa} - \left(\frac{ds}{dt}\right)^2;$$

$$\therefore \kappa\, dt = \frac{d\left(\dfrac{ds}{dt}\right)}{\dfrac{g}{\kappa} - \left(\dfrac{ds}{dt}\right)^2}.$$

Separating this into two partial fractions, and integrating, gives

$$\kappa t = \tfrac{1}{2}\left(\frac{\kappa}{g}\right)^{\frac{1}{2}} \log \frac{g^{\frac{1}{2}} + \kappa^{\frac{1}{2}}\dfrac{ds}{dt}}{g^{\frac{1}{2}} - \kappa^{\frac{1}{2}}\dfrac{ds}{dt}}.$$

Passing to exponentials gives

$$v = \frac{ds}{dt} = \left(\frac{g}{\kappa}\right)^{\frac{1}{2}} \frac{e^{2t(\kappa g)^{\frac{1}{2}}} - 1}{e^{2t(\kappa g)^{\frac{1}{2}}} + 1}; \qquad (b)$$

which gives the velocity in terms of the time. To find it in terms of the space, multiply equation (*a*) by ds and put it under the form

$$\frac{d\left(\dfrac{ds}{dt}\right)^2}{\dfrac{g}{\kappa} - \left(\dfrac{ds}{dt}\right)^2} = 2\kappa\, ds.$$

Proceeding as before, observing the proper limits, we find

$$2\kappa s = -\log \frac{g - \kappa\left(\dfrac{ds}{dt}\right)^2}{g};$$

$$\therefore \frac{ds}{dt} = v = \sqrt{\frac{g}{\kappa}\left(1 - e^{-2\kappa s}\right)}. \qquad (c)$$

If $s = \infty$, $v = \sqrt{\dfrac{g}{\kappa}}$, and hence the velocity tends towards a constant.

From equation (*b*), multiplying the terms of the fraction by $e^{-t(\kappa g)^{\frac{1}{2}}}$, and observing that the numerator becomes the differential of the denominator integrating, and passing to exponentials, we have,

$$2\, e^{\kappa s} = e^{t(\kappa g)^{\frac{1}{2}}} + e^{-t(\kappa g)^{\frac{1}{2}}}; \qquad (d)$$

which gives the space in terms of the time.

A neat solution of equation (*a*) may be found by Lagrange's method of *Variation of Parameters.*

30*th*. *Suppose that the body is projected upward in the air, having the same coefficient of resistance as in the preceding example.*

Take the origin at the point of propulsion, u being the initial velocity; then

$$\frac{d^2s}{dt^2} \text{ or, } \frac{d}{dt}\frac{ds}{dt} = -g - \kappa\left(\frac{ds}{dt}\right)^2; \qquad (e)$$

hence,
$$\kappa dt = -\frac{d\left(\frac{ds}{dt}\right)}{\frac{g}{\kappa} + \left(\frac{ds}{dt}\right)^2};$$

$$\therefore \kappa t = -\left(\frac{\kappa}{g}\right)^{\frac{1}{2}}\left\{\text{tang}^{-1}\left(\frac{\kappa}{g}\right)^{\frac{1}{2}}\left(\frac{ds}{dt}\right) - \text{tang}^{-1}\left(\frac{\kappa}{g}\right)^{\frac{1}{2}} u\right\}.$$

Solving this equation for $\frac{ds}{dt}$; we have,

$$v = \frac{ds}{dt} = \left(\frac{g}{\kappa}\right)^{\frac{1}{2}} \frac{u\sqrt{\kappa} - \sqrt{g}\tan t\sqrt{\kappa g}}{\sqrt{g} + u\sqrt{\kappa}\tan t\sqrt{\kappa g}}. \qquad (f)$$

Substitute $\sin t\sqrt{\kappa g} \div \cos t\sqrt{\kappa g}$ for $\tan t\sqrt{\kappa g}$ and the numerator becomes the differential of the denominator, and observing that $t = 0$ for $s = 0$, we have

$$s = \frac{1}{\kappa}\log\frac{u\sqrt{\kappa}\sin t\sqrt{\kappa g} + \sqrt{g}\cos t\sqrt{\kappa g}}{\sqrt{g}};$$

which gives the space in terms of the time.

Multiply equation (e) by ds and it may be put under the form

$$2\kappa ds = -\frac{d\left(\frac{ds}{dt}\right)^2}{\frac{g}{\kappa} + \left(\frac{ds}{dt}\right)^2}.$$

Integrating, observing that u is the initial velocity, and

$$2\kappa s = -\log\frac{g + \kappa\left(\frac{ds}{dt}\right)^2}{g + \kappa u^2};$$

$$\therefore \frac{ds}{dt} = v = \left(u^2 e^{-2\kappa s} - \frac{g}{\kappa}\left(1 - e^{-2\kappa s}\right) \right)^{\frac{1}{2}}. \qquad (g)$$

At the highest point $v = 0$, which in (f) and (g) gives

$$t = (\kappa g)^{-\frac{1}{2}} \tan^{-1} u \left(\frac{\kappa}{g}\right)^{\frac{1}{2}}; \qquad (h)$$

and,

$$s = \frac{1}{2\kappa} \log \left(1 + \frac{\kappa}{g} u^2\right). \qquad (i)$$

Substitute this value of s in equation (c) of the preceding example, and we have

$$v = \sqrt{\frac{g}{\kappa}\left(1 - \frac{1}{(1 + \frac{\kappa}{g} u^2)}\right)}$$

$$= \sqrt{\frac{g}{g + \kappa u^2}}\, u\,;$$

which gives the velocity in descending to the point from which it started; and as it is less than u, the velocity of return will be less than that with which it was thrown upward. This is because the resistance of the air is against the velocity during the entire movement, both upwards and downwards.

The same value of s (Eq. (i)) substituted in (d) of the preceding example gives the time of descent,

$$t = \frac{1}{2\sqrt{\kappa g}} \log \frac{\sqrt{g + \kappa u^2} + u\sqrt{\kappa}}{\sqrt{g + \kappa u^2} - u\sqrt{\kappa}};$$

which differs from the time of the ascent, as given by (h) above.

31*st*. *Suppose that the force is attractive and varies inversely as the cube of the distance, and that the medium resists as the square of the velocity, and as the square of the density, the density varying inversely as the distance from the origin.*

Let $\kappa =$ the coefficient of resistance, being the resistance for a unit of density of the medium and a unit of velocity

then $\frac{\kappa}{s^2}\left(\frac{ds}{dt}\right)^2$ = the resistance at any point.

$$\therefore \frac{d^2s}{dt^2} = -\frac{\mu}{s^3} + \frac{\kappa}{s^2}\left(\frac{ds}{dt}\right)^2.$$

Multiply by $2ds$, and we have

$$d\left(\frac{ds}{dt}\right)^2 - \frac{2\kappa}{s^2}\left(\frac{ds}{dt}\right)^2 ds = -\frac{2\mu}{s^3}\,ds.$$

This is a linear differential equation of which the integrating factor is $e^{\frac{2\kappa}{s}}$. The initial values are $t = 0$, and $s = a$ for $v = u$;

$$\therefore e^{\frac{2\kappa}{s}}\left(\frac{ds}{dt}\right)^2 - e^{\frac{2\kappa}{a}}u^2 = \frac{\mu}{2\kappa^2}\left\{\frac{2\kappa - s}{s}\,e^{\frac{2\kappa}{s}} - \frac{2\kappa - a}{a}\,e^{\frac{2\kappa}{a}}\right\},$$

which gives the velocity in terms of the space. The final integral cannot be found.

25. Work and Vis Viva (*or living force.*)—Resuming equation (21), and multiplying both members by ds, we have

$$Fds = M\frac{d^2s}{dt^2}ds.$$

Integrating between the limits, $v = v_0$ for $s = 0$; and $v = v$ for $s = s$, we have

$$\int Fds = \tfrac{1}{2}M(v^2 - v_0^2). \qquad (23)$$

If $v_0 = 0$, we have

$$\int Fds = \tfrac{1}{2}Mv^2. \qquad (24)$$

The expression $\frac{1}{2}Mv^2$ is called the vis viva (or *living force*) of a body whose mass is M and velocity v. Its physical importance is determined from the first member of the equation, which is called the work done by a force F in the space s. Hence *the vis viva equals the work done by the moving force.*

Work, *mechanically, is overcoming resistance.* It requires a certain amount of work to raise one pound one foot, and twice

that amount to raise two pounds one foot, or one pound two feet. Similarly, if it requires 100 pounds to move a load on a horizontal plane, a certain amount of work will be accomplished in moving it one foot, twice that amount in moving it two feet, and so on. Hence, generally, if

$F =$ a constant force which overcomes a constant resistance, and

$s =$ the space over which F acts projected on the action-line of the force, then

$$Work = U = Fs; \qquad (25)$$

and similarly, if

$$F = a\ variable\ force, \text{ then}$$
$$Work = U = \Sigma Fds; \qquad (26)$$

and if F is a function of s we have

$$U = \int Fds.$$

The UNIT of work is one pound raised vertically one foot.

The total work, according to equation (25), is independent of the time, since the space may be accomplished in a longer or shorter time.

But implicitly it is a function of the *time* and *velocity*. If the work be done at a uniform rate, we have

$$s = vt,$$

and

$$Fs = Fvt.$$

If $t = 1$, we have

$$Fv, \qquad (28)$$

which is called the *Dynamic Effect*, or *Mechanical Power*.

MECHANICAL POWER *is the rate of doing work*. It is measured by the amount of work done, or which the agent is capable of doing, in a unit of time when working uniformly. *The unit* most commonly employed is called the *horse-power*, which equals 33,000 pounds raised one foot per minute.

Every moving body on the surface of the earth does work, for it overcomes a resistance, whether it be friction or resistance of the air, or some other resistance. The same is true of every body in the universe, unless it moves

in a non-resisting medium.* Animals work not only as beasts of burden, but in their sports and efforts to maintain life; water as it courses the stream wears its banks or the bed, or turns machinery; wind fills the sail and drives the vessel, or turns the windmill, or in the fury of the tornado levels the forest, and often destroys the *works* of man. The raising of water into the air by means of evaporation; the wearing down of hills and mountains by the operations of nature; the destruction which follows the lightning-stroke, etc., are examples of work.

Work may be useful or prejudicial. That work is useful which is directly instrumental in producing useful effects, and prejudicial when it wears the machinery which produces it. Thus in drawing a train of cars, the useful work is performed in moving the train, but the prejudicial work is overcoming the friction of the axles, the friction on the track, the resistance of the air, the resistance of gravity on up grades, etc. It is not always possible to draw a practical line between the useful and prejudicial works, but the sum of the two always equals the total work done, and hence for economy the latter should be reduced as much as possible.

In order to determine practically the work done, the intensity of the force and the space over which it acts must be measured simultaneously. Some form of spring balance is commonly used to measure the force, and when thus employed is called a Dynamometer. It is placed between the moving force and the resistance, and the reading may be observed, or autographically registered by means of suitable mechanism. The corresponding space may also be measured directly, or secured automatically. There are many devices for securing these ends, and not a few make both records automatically and simultaneously.

If the force is not a continuous function of the space, equation (26) must be used. The result may be shown graphically by laying off on the abscissa, AB, the distances ac, ce, etc., proportional to the spaces, and erecting ordinates ab, cd, ef, etc., proportional to the corresponding forces, and joining their upper ends by a broken line, or, what is better, by a line which

* All space is filled with something, since light is transmitted from all directions. But is it not possible that there may be a *something* through which *bodies* may move without resistance?

is slightly curved, the amount and direction of curvature being *indicated* by the broken line previously constructed; and the area thus inclosed will *represent* the work. The area will be given by the formula $\Sigma F . \overline{\Delta s}$.

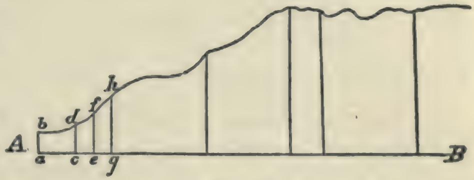

FIG. 16.

Simpson's rule for determining the area is:—

Divide the abscissa AB into an even number of equal parts, erect ordinates at the points of division, and number them in the order of the natural numbers. Add together four times the even ordinates, twice the odd ordinates and the extreme ordinates, and multiply the sum by one third of the distance between any two consecutive ordinates.

If y_1, y_2, y_3, etc., are the successive ordinates, and l the distance between any two consecutive ones, the rule is expressed algebraically as follows:—

$$Area = \tfrac{1}{3}l\,(y_1 + 4y_2 + 2y_3 + 4y_4 \; - \; - \; - + y_n). \qquad (29)$$

If the applied pressure, F, is exerted against a body which is perfectly free to move, generating a velocity v, then the work which has been expended is, equation (24), $\frac{1}{2}Mv^2$. This is called *stored work*, and the amount of work which will be done by the moving body in being brought to rest will be the same amount. If the body is not perfectly free the quantity $\frac{1}{2}Mv^2$ is the quantity of work which has been expended by so much of the applied force as exceeds that which is necessary in overcoming the frictional resistance. Thus a locomotive starts a train from rest, and when the velocity is small the power exerted by the locomotive may exceed considerably the resistances of friction, air, etc., and produce an increasing velocity, until the resistances equal constantly the tractive force of the locomotive, after which the velocity will be uniform. The work done by the locomotive in producing the velocity v in excess of that done in overcoming the resistances will be $\frac{1}{2}Mv^2$, in which M is the mass of the train, including the locomotive.

We see that double the velocity produces four times the work. This is because twice the force produces twice the velocity, and hence the body will pass over twice the space in the same time, so that in producing double the velocity we have $2F.2s = 4Fs$, and similarly for other velocities.

[We have no single word to express the unit of living force. If a unit of mass moving with a velocity of one foot per second be the unit of living force, and be called a *Dynam*, then would the living force for any velocity and mass be a certain number of *Dynams*.]

Since *work is not force*, but the effect of a force exerted through a certain space, independently of the time, we call it, for the sake of brevity, *space-effect*.

Vis viva, or living force, is not force, but it equals the work stored in a moving mass. It equals the *space-effect*.

[The expression Mv^2 was called the vis viva in the first edition of this work, and is still so defined by many writers; but there appears to be a growing tendency towards the general adoption of the definition given in the text. It is immaterial which is used, provided it is always used in the same sense.]

EXAMPLES.

1. A body whose weight is 10 pounds is moving with a velocity of 25 feet per second; required the amount of work which will be done in bringing it to rest.

Ans. 97.2 foot-pounds.

2. A body falls by the force of gravity through a height of h feet; required the work stored in it.

Let $W =$ the weight of the body,
$M =$ the mass of the body,
$g =$ acceleration due to gravity, and
$v =$ the final velocity, then
$v^2 = 2gh$, and $Mg = W$;

$$\therefore \tfrac{1}{2}Mv^2 = \frac{W}{2g} \cdot 2gh = Wh.$$

3. A body whose weight is 100 pounds is moving on a horizontal plane with a velocity of 15 feet per second; how far will it go before it is brought to rest, if the friction is constantly 10 lbs?

Ans. $= 34.9 +$ ft.

4. A hammer whose weight is 2000 pounds has a velocity of 20 feet per second; how far will it drive a pile if the constant resistance is 10,000 pounds, supposing that the whole *vis viva* is expended in driving the pile?

5. If a train of cars whose weight is 100,000 pounds is moving with a velocity of 40 miles per hour, how far will it move before it is brought to rest by the force of friction, the friction being 8 pounds per ton, or $\frac{8}{2000}$ of the total weight?

6. If a train of cars weighs 300 tons, and the frictional resistance to its movement is 8 pounds per ton; required the horse-power which is necessary to overcome this resistance at the rate of 40 miles per hour.

Ans. 256.

7. If the area of a steam piston is 75 square inches, and the steam pressure is 60 pounds per square inch, and the velocity of the piston is 200 feet per minute, required the horse-power developed by the steam.

8. If a stream of water passes over a dam and falls through a vertical height of 16 feet, and the transverse section of the stream at the foot of the fall is one square foot, required the horse-power that is constantly developed.

Let $g = 32\frac{1}{6}$ feet, and the weight of a cubic foot of water, $62\frac{1}{2}$ lbs.

Ans. 58.2 +

9. A steam hammer falls vertically through a height of 3 feet under the action of its own weight and a steam pressure of 1000 pounds. If the weight of the hammer is 500 pounds, required the amount of work which it can do at the end of the fall.

26. Energy *is the capacity of an agent for doing work.* The energy of a moving body is called *actual* or *Kinetic energy*, and is expressed by $\frac{1}{2}Mv^2$. But bodies not in motion may have

a capacity for work when the restraining forces are removed. Thus a spring under strain, water stored in a mill-dam, steam in a boiler, bodies supported at an elevation, etc., are examples of stored work which is *latent*. This is called *Potential energy*. A *moving* body may possess potential energy entirely distinct from the actual. Thus, a locomotive boiler containing steam, may be moved on a track, and the kinetic energy would be expressed by $\frac{1}{2}Mv^2$, in which M is the mass of the boiler, but the *potential* energy would be the amount of work which the steam is capable of doing when used to run machinery, or is otherwise *employed*. These principles have been generalized into a law called the *Conservation of energy*, which implies that the total energy, including both Kinetic and Potential, in the universe remains constant. It is made the fundamental theorem of modern physical science.

The energy stored in a moving body is not changed by changing the direction of its path, *provided* the velocity is not changed; for its energy will be constantly expressed by $\frac{1}{2}Mv^2$. Such a change may be secured by a force acting continually normal to the path of the moving body; and hence we say *that a force which acts continually perpendicular to the path of a moving body does no work upon the body*. Thus, if a body is secured to a point by a cord so that it is compelled to move in the circumference of a circle; the tension of the string does no work, and the vis viva is not affected by the body being constantly deflected from a rectilinear path.

MOMENTUM.

27. Resuming again equation (21), multiplying by dt, and integrating gives,

$$\int_0^t Fdt = M\int\frac{d^2s}{dt} = M\frac{ds}{dt} = Mv. \qquad (30)$$

The expression Mv is called *momentum*, and by comparing it with the first member of the equation we see that it is the effect of the force F acting during the time t, and is independent of the space. For the sake of brevity we may call the momentum a *time-effect*.

If the body has an initial velocity we have

$$\int_{t_0}^{t} Fdt = M(v - v_0); \qquad (31)$$

which is the momentum gained or lost in passing from a velocity v_0 to v.

Momentum is sometimes called *quantity of motion*, on account of its analogy to some other quantities. Thus the intensity of heat depends upon temperature, and is measured in degrees; but the quantity of heat depends upon the volume of the body containing the heat and its intensity. The intensity of light may be uniform over a given surface, and will be measured by the light on a unit of surface; but the quantity is the product of the area multiplied by the intensity. The intensity of gravity is measured by the acceleration which is produced in a falling body, and is independent of the mass of the body; but the quantity of gravity (or total force) is the product of the mass by the intensity (or Mg). Similarly with momentum. The *velocity* represents the *intensity* of the motion, and is independent of the mass of the body; but the quantity of motion is the product of the mass multiplied by the velocity.

Differentiating (30) and reducing, gives

$$F = M\frac{dv}{dt};$$

which is the same as (18), and in which $\frac{dv}{dt}$ is a velocity-increment; *hence the rate of change of the momentum per unit of time is a measure of the force which is acting on the body.*

If the force F is constant we have from (30),

$$Ft = Mv;$$

and for another force F' acting during the same time

$$F't = M'v';$$

$$\therefore F : F' :: Mv : M'v';$$

hence, the forces are directly as the momenta produced by them respectively.

If the forces are variable, let

$$\int_0^t Fdt = Q = Mv, \text{ and } \int_0^{t'} F'dt = Q' = M'v';$$

then
$$Q : Q' :: Mv : M'v';$$

hence the *time-effects* are directly as the momenta impressed.

We thus have several distinct quantities growing out of equation (21) of which the English units are as follows :—

The unit of force, F, is..........................	1 lb.
The unit of work or space effect is	1 lb × 1 ft.
The unit of vis viva is..............	1 lb of mass × 1^2 ft. × 1 sec.
The unit of momentum............	1 lb of mass × 1 ft. × 1 sec.

IMPULSE.

28. *An impulse* is the *effect of a blow.* When one body strikes another, an impact is said to take place, and certain effects are produced upon the bodies. These *effects* are produced in an exceedingly short time, and for this reason they are sometimes called *instantaneous forces;* which, being strictly defined, means a force which produces its effect *instantly, requiring no time for its action;* but no such force exists in nature. The law of action during impact is not generally known, but it must be some function of the time.

Resuming equation (31), we have

$$\int_{t_0}^t Fdt = M(v - v_0);$$

which is true, whatever be the relation between the force F and the time t. If the initial velocity of the body be zero, we have

$$v_0 = 0,$$

and
$$\int_{t_0}^t Fdt = MV = Q.$$

We see from the above equation that as t diminishes F must increase to produce the same *effect.* We see that in this

case the first member is the *time-effect* of an impulse, and the second member measures its effect in producing a change of velocity. Calling this value Q, we have

$$Q = M(v - v_0) = MV. \qquad (31a)$$

Hence, the measure of an impulse in producing a change of velocity of a body is the increased (or decreased) momentum produced in the body.

This is the same as when the force and time are finite. If the force were *strictly* instantaneous the velocity would be changed from v_0 to v without moving the body, since it would have no time in which to move it.

Similarly from equation (23) we have

$$\int_0^s Fds = \tfrac{1}{2}M(v^2 - v_0^2);$$

in which for an impulse F will be indefinitely large; and hence *the work done by an impulse is measured in the same way as for finite forces.*

All the *effects* therefore of an impulse are measured in the same way as the *total effects* produced by a finite force.

In regard to *forces*, we investigate their laws of action; or having those laws and the initial condition of the body we may determine the velocity, energy, or position of the body at any instant of time or at any point in space, and hence we may determine final results; but in regard to *impulses* we determine only certain final results without assuming to know anything of the laws of action of the forces, or of the time or space occupied in producing the effect.

The terms "*Impulsive force,*" and "*Instantaneous force,*" are frequently used to denote the effect of an *Impact;* but since the effect is not a force, they are ambiguous, and the term *Impulse* appears to be more appropriate.

An *incessant force* may be considered as the action of an infinite number of infinitesimal impulses in a finite time.

The question is sometimes asked, "What is the force of a

blow of a hammer?" If by *the force* is meant the pressure in pounds between the face of the hammer and the object struck, it cannot be determined unless the law of resistance to compression between the bodies is known during the contact of the bodies. But this law is generally unknown. The pressure begins with nothing at the instant of contact and increases very rapidly up to the instant of greatest compression, after which the pressure diminishes. The pressure involves the elasticity of both bodies; the rapidity with which the force is transmitted from one particle to another; the amount of the distortion; the pliability of the bodies; the duration of the impact; and some of these depend upon the degree of fixedness of the body struck; and several other minor conditions; and hence we consider it impossible to tell *exactly* what the *force* is.

Examples.

1. Two bodies whose weights are W and W_1 are placed very near each other, and an explosive is discharged between them; required the relative velocities after the discharge.

2. A man stands upon a rough board which is on a perfectly smooth plane, and jumps off from the board; required the relative velocities of the man and board.

[Obs. The common centre of gravity of the man and board will remain the same after they separate that it was before. After separating they would move on forever if they did not meet with any obstacle to prevent their motion.]

3. A man whose weight is 150 pounds walks from one end of a rough board to the other, which is twelve feet long, and free to slide on a perfectly smooth plane; if the board weighs 50 pounds, required the distance travelled by the man in space.

4. In example 3 of article 24, suppose that the weight 10 pounds is permitted to fall freely through a height h, when it produces an impulse on the body (50 pounds) through the intermediate inextensible string; required the initial velocity of the body.

Let $v_0 = \sqrt{2gh}$ = the velocity of the weight just before the impulse; and

v = the velocity immediately afterward, which will be the common velocity of the body and weight;

then

$$Q = \frac{50}{g} v = \frac{10}{g}(v_0 - v);$$

$$\therefore v = \tfrac{1}{6} v_0.$$

The subsequent motion may be found by equation (21), observing that the initial velocity is v.

The tension on the string will be infinite if it is inextensible, but practically it will be finite, for it will be more or less elastic.

[Some writers have used the expression *impulsive tension* of the string instead of *momentum.*]

5. If a shell is moving in a straight line, in vacuo, with a velocity v, and bursts, dividing into two parts, one part moving directly in advance with double the velocity of the body; what must be the ratio of the weights of the two parts so that the other part will be at rest after the body bursts?

6. Explain how a person sitting in a chair may move across a room by a series of jerks without touching the floor. (Can he advance if the floor is perfectly smooth?)

7. A person is placed on a perfectly smooth plane, show how he can get off if he cannot reach the edge of the plane.

The same impulse applied to a small body will impart a greater amount of energy than if applied to a large one. Thus, in the discharge of a gun, the impulse imparted to the gun equals that imparted to the ball, but the work, or destructive effect, of the gun is small compared with that of the ball. The *time of the action* of the explosive is the same upon both bodies, but the space moved over by the gun will be small compared with that of the ball during that time.

The product Mv, being the same for both, as M decreases v increases, but the work varies as the square of the velocity.

DIRECT CENTRAL IMPACT.

29. If two bodies impinge upon one another, so that the line of motion before impact passes through the centre of the bodies, it is said to be *central;* and if at the same time the common tangent at the point of contact is perpendicular to the line of motion, it is said to be *direct* and *central.* If their common tangent is perpendicular to the line of motion, but if the latter does not pass through the centre of the body impinged upon, it is called *eccentric impact.* In this place, we consider only the simplest case; that in which the impact is *direct* and *central.*

When two bodies impinge directly against one another, whether moving in the same or opposite directions, they mutually displace the particles in the vicinity of the point of contact, producing compression which goes on increasing until it becomes a maximum, at which instant they have a common velocity. A complete analysis of the motion during contact involves a knowledge of the motion of all the particles of the mass, and would make an exceedingly complicated problem, but the motion at the instant of maximum compression may be easily found if we assume that the compression is instantly distributed throughout the mass.

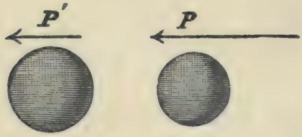

FIG. 17.

Let M_1 and M_2 be the respective masses of the bodies;

v_1 and v_2 the respective velocities before impact, both positive and $v_1 > v_2$;

v_1' and v_2' the respective velocities at the instant of maximum compression, $v'_2 > v_2$, and

Q_1 and Q_2 the momenta gained respectively by the bodies during compression.

Then from (31)

$$Q_2 = M_2 (v_2' - v_2),$$

which will be the momentum gained by M_2 on account of the action of M_1.

Similarly

$$Q_1 = M_1 (v_1' - v_1),$$

which, being negative, will be the momentum lost by M_1 on account of the action of M_2.

But at the instant of greatest compression

$$v_1' = v_2';$$

and, because they are in mutual contact during the same time, their *time-effects* are equal, but in opposite directions,

$$\therefore\ Q_1 = -Q_2.$$

Combining these four equations, we find by elimination

$$-Q_1 = \frac{M_1 M_2}{M_1 + M_2}(v_1 - v_2) = Q_2 \qquad (32)$$

$$v_1' = \frac{M_1 v_1 + M_2 v_2}{M_1 + M_2} = v_2'; \qquad (33)$$

which velocity remains constant for perfectly non-elastic bodies after impact, since such bodies have no power of restitution and will move on with a common velocity.

DIRECT CENTRAL IMPACT OF ELASTIC BODIES.

30. Elastic bodies are such as regain a part or all of their distortion when the distorting force is removed. If they regain their original form they are called *perfectly elastic*, but if only a part, they are called *imperfectly elastic.* After the impact has produced a maximum compression, the elastic force of the bodies causes them to separate, but all the effect which the force of restitution can produce upon the movement of the bodies, evidently takes place while they are in contact. If they are perfectly elastic and do not fully regain their form at the instant of separation, they will continue to regain their form after separation, but the latter effect we do not consider in this place. The ratio between the forces of compression and those of restitution has often been called the *modulus of elasticity*, but as some ambiguity results from this definition, we will call it the *modulus of restitution.* At every point of the restitution there is assumed to be a constant ratio between the force due to compression and that to restitution. But it is unnecessary for present purposes to *trace* these effects, for by equation (31) we may determine the result when the bodies finally separate from each other.

Let e_1 = the ratio of the force of compression to that of restitution of one body, which is called the *modulus of restitution.*

e_2 = the corresponding value for the other;

V_1 = the velocity of M_1 at the instant when they separate from each other; and

V_2 = the corresponding velocity for M_2.

Then from equation (31)

$$e_1 Q_1 = M_1 (V_1 - v_1'); \qquad (34)$$

$$e_2 Q_2 = M_2 (V_2 - v_2'). \qquad (35)$$

As before $Q_1 = - Q_2$ and we will also assume that $e_1 = e_2 = e$. These combined with (32) and (33) give

$$V_1 = \frac{M_1 v_1 + M_2 v_2}{M_1 + M_2} - \frac{eM_2}{M_1 + M_2}(v_1 - v_2); \qquad (36)$$

$$V_2 = \frac{M_1 v_1 + M_2 v_2}{M_1 + M_2} + \frac{eM_1}{M_1 + M_2}(v_1 - v_2). \qquad (37)$$

31. Discussion of Equations (36) and (37).

1°. If the bodies are perfectly non-elastic, $e = 0$.

$$\therefore V_1 = \frac{M_1 v_1 + M_2 v_2}{M_1 + M_2} = V_2; \qquad (38)$$

which is the same as (33).

2°. If the restitution is perfect $e = 1$.

$$\therefore V_1 = v_1 - \frac{2M_2}{M_1 + M_2}(v_1 - v_2); \qquad (39)$$

$$V_2 = v_2 + \frac{2M_1}{M_1 + M_2}(v_1 - v_2). \qquad (40)$$

From (38) we have

$$V_1 - v_1 = -\frac{M_2}{M_1 + M_2}(v_1 - v_2);$$

and,
$$V_2 - v_2 = \frac{M_1}{M_1 + M_2}(v_1 - v_2).$$

Similarly from (39) and (40)

$$V_1 - v_1 = -\frac{2M_2}{M_1 + M_2}(v_1 - v_2);$$

$$V_2 - v_2 = \frac{2M_1}{M_1 + M_2}(v_1 - v_2);$$

hence, the velocity lost by one body and gained by the other is twice as much when the bodies are perfectly elastic as when they are perfectly non-elastic.

3°. If $M_1 = M_2$, then for perfect restitution we have

$$V_1 = v_1 - \frac{2M_1}{M_1 + M_1}(v_1 - v_2) = v_2;$$

$$V_2 = v_2 + \frac{2M_1}{M_1 + M_1}(v_1 - v_2) = v_1;$$

that is, they will interchange velocities.

4°. If M_1 impinges against a fixed body, we have $M_2 = \infty$, and $v_2 = 0$.

$$\therefore\ V_1 = -e\,v_1.$$

This furnishes a convenient mode of determining e. For if a body falls from a height h upon a fixed horizontal plane, it will rebound to a height h_1;

$$\therefore h_1 = e^2 h, \text{ or } e = \sqrt{\frac{h_1}{h}}.$$

Also if $e = 1$

$$V_1 = -v_1;$$

or the velocity after impact will be the same as before, but in an opposite direction.

Also if $e = 0$, $V_1 = 0$; or the velocity will be destroyed.

5°. If $v_2 = 0$ we have

$$\left.\begin{aligned} V_1 &= \frac{M_1 - eM_2}{M_1 + M_2} v_1; \\ V_2 &= \frac{M_1 + eM_1}{M_1 + M_2} v_1. \end{aligned}\right\} \quad (41)$$

Examples.

(1.) A mass M_1 with a velocity of 10, impinges on M_2 moving in an opposite direction, moving with a velocity 4 and has its velocity reduced to 5; required the relative magnitudes of M_1 and M_2.

(2.) Two inelastic bodies, weighing 8 and 5 pounds respectively, move in the same direction with velocities 7 and 3; required the common velocity after impact, and the velocity lost and gained by each.

(3.) If M_1 weighs 12 pounds and moves with a velocity of 15, and is impinged upon by a body M_2 weighing 16 pounds, producing a common velocity of 30, required the velocity of M_2 before impact if it moves in the same or opposite direction.

(4.) If $5M_1 = 6M_2$, $6v_1 = -5v_2$, $v_2 = 7$, and $e = \frac{2}{3}$; required the velocity of each after impact.

(5.) If $M_1 = 2M_2$, $V_1 = \frac{2}{3}v_1$, and $v_2 = 0$; required e.

(6.) If v_1 is 26, M_2 is moving in an opposite direction with a velocity of 16; $M_1 = 2M_2$, $e = \frac{2}{3}$; required the distance between them $5\frac{1}{2}$ seconds after impact.

(7.) Two bodies are perfectly elastic and move in opposite directions; the weight of M_1 is twice M_2, but $v_2 = 2v_1$; required the velocities after impact.

(8.) There is a row of perfectly elastic bodies in geometrical progression whose common ratio is 3, the first impinges on the second, the second on the third and so on; the last moves off with $\frac{1}{64}$ the velocity of the first. What is the number of bodies?

Ans. 7.

LOSS OF VIS VIVA IN THE IMPACT OF BODIES.

32. Before impact the vis viva of both bodies was

$$\tfrac{1}{2}M_1v_1^2 + \tfrac{1}{2}M_2v_2^2;$$

and after impact

$$\tfrac{1}{2}M_1 V_1^2 + \tfrac{1}{2}M_2 V_2^2;$$

which by means of (36) and (37) becomes

$$M_1 V_1^2 + M_2 V_2^2 = M_1 v_1^2 + M_2 v_2^2 - \frac{(1-e^2)\, M_1 M_2}{M_1 + M_2}(v_1 - v_2)^2. \quad (42)$$

For perfectly elastic bodies $e = 1$ and the last term disappears; *hence in the impact of perfectly elastic bodies no vis viva is lost.*

If the bodies are imperfectly elastic e is less than 1, and since $(v_1 - v_2)^2$ is always positive, it follows that *in the impact of imperfectly elastic bodies vis viva is always lost, and the greatest loss is suffered when the bodies are perfectly non-elastic.*

If $e = 0$, (42) becomes

$$M_1(v_1^2 - V_1^2) + M_2(v_2^2 - V_2^2) = \frac{M_1 M_2}{M_1 + M_2}(v_1 - v_2)^2; \quad (43)$$

in which each member is the total loss by both bodies. It is also the loss up to the instant of greatest compression when the bodies are elastic.

If M_2 is very large compared with M_1 we have from (38)

$$V_1 = v_2 \text{ nearly}, = V_2,$$

and (43) becomes

$$M_1 v_1^2 - M_1 V_1^2 = M_1(v_1 - V_1)^2,$$

the second member of which is frequently used in hydraulics for finding the *vis viva* lost by a sudden change of velocity.

THESE INVESTIGATIONS show the great utility of springs in vehicles and machines which are subjected to impact.

RELATIONS OF FORCE, MOMENTUM, WORK, AND VIS VIVA.

33. WE MAY NOW DETERMINE THE EXACT OFFICE in the same problem of the quantities ;—*force, momentum, work, and vis viva.* Suppose that a force, whether variable or constant, impels a body, it will in a time t generate in the mass M a certain velocity v. This *force* may at any instant of its action be measured by a certain number of pounds or its equivalent.

Suppose that this mass impinges upon another body, which may be at rest or in motion. In order to determine the effect upon their *velocities* we use the principle of *momentum*, as has been shown. But the bodies are compressed during impact and hence *work* is done. The amount of work which they are capable of doing is equal to the sum of their *vis viva*; and if they are brought to rest all this work is expended in compressing them. If the velocity of a body after impact is less than that before, it has done an amount of work represented by $\frac{1}{2}M(v^2 - V^2)$, and similarly if the other body has its velocity increased kinetic energy is imparted to it. *The distortions of bodies represent a certain amount of work expended.* And this explains why in the impact of imperfectly elastic bodies *vis viva* is always lost, for a portion of the distortion remains. But no force is lost. One of the grandest generalizations of physical science is, *that no energy in nature is lost.* In the case of impact, compression develops heat, and this passes into the air or surrounding objects, and the amount of energy which is stored in the heat, electricity or other element or elements, which is developed by the compression, exactly equals that lost to the masses. We thus see that in the case of moving bodies, *force impels, momentum determines velocity after impact, and work or vis viva represents the resistance which the particles offer to being displaced.*

34. Statics is that case in which the force or forces which would produce motion are instantly arrested, resulting in pressure only. The expression for the elementary work which a force can do is Fds, but if the space vanishes, we have, $Fds = 0$. This, as we shall see hereafter, is a special case of "virtual velocities."

The forces which act upon a body may be in equilibrium and yet motion exist, but in such cases the velocity is uniform.

35. The term *power* is often used in the same sense as *force*, but generally it refers to an acting agent. The term *mechanical power* is not only recognized in this science, but has a specific meaning, and for the purpose of avoiding ambiguity, it is better to use the term *effort* in reference to mechanical agents. Thus, instead of saying *the power and weight*, as is often done, say *the effort and resistance.*

36. INERTIA *implies passiveness or want of power.* It means that matter has no power within itself to put itself in motion, or when in motion to change its rate of motion. Unless an external force be applied to it, it would, if at rest, remain forever in that condition; or if in motion, continue forever in motion. Gravity, which is a force apparently inherent in matter, can produce motion only by its action upon other matter.

INERTIA *is not a force*, but because of the property above explained, those impressed forces which produce motion are measured by the product of the mass into the acceleration as explained in preceding articles; and many writers call this MEASURE *the force of inertia.*

37. NEWTON'S THREE LAWS OF MOTION

Sir Isaac Newton expressed the fundamental principles of motion in the form of three laws or mechanical axioms; as follows:—

1st. Every body continues in its state of rest or of uniform motion in a straight line unless acted upon by some external force.

2d. Change of motion is proportional to the force impressed, and is in the *direction* of the line in which the force acts.

3d. To every action there is opposed an equal reaction.

[As simple as these laws appear to the student of the present day, the science of Mechanics made no essential progress until they were recognized. See Whewell's *Inductive Sciences*, 3d ed., vol. 1, p. 311.]

38. In all the problems thus far considered, it has been assumed that the action-line of the force or forces passed through the centre of the mass, producing a motion of translation only. But if the action-line does not pass through the centre, it will produce both translation and rotation.

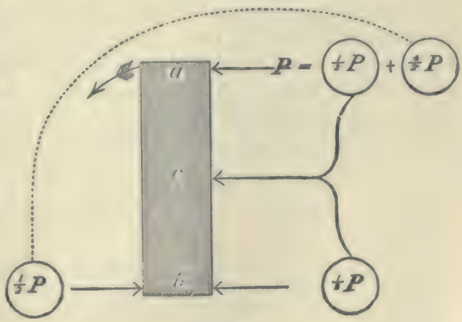

FIG. 18.

In eccentric impact both translation and rotation is produced. The centre of the body will move in a straight line, but every other point will describe arcs of circles in reference to the centre of the body, which in space will be curves more or less elongated. The velocity of translation will be directly proportional to the intensity of the impulse imparted to the body, but the angular velocity will depend upon the intensity of the impulse and the distance of the point of impact from the centre of the body.

In Figure 18, let $Q = Mv$ be the impulse imparted to the body; in which M is the mass of the body and v the velocity of the centre. Let this impulse be imparted at a. At b, a distance from the centre $= cb = ac$, let two equal and opposite impulses be imparted, each equal to $\frac{1}{2}Q$. The impulse Q, equals $\frac{1}{2}Q + \frac{1}{2}Q$. The four impulses evidently produce the same effect upon the body as the single impulse Q. If now one of the impulses, $\frac{1}{2}Q$, above the centre is combined with the equal and parallel one acting in the same direction below the centre, their effect will be equivalent to a single one, equal to Q applied at the centre c. This produces translation only. The other $\frac{1}{2}Q$ above the centre combined with the equal and opposite $\frac{1}{2}Q$ below the centre, produces rotation only; and it is evident that the greater the distance a, the point of impact, is from the centre, the greater will be the amount of rotation.

An impact (or blow) at a to produce a velocity v at the centre of the body, must act through a greater space during contact, or the impacting body must move with a greater velocity, than if the impact be in a line passing through the centre c.

(The entire energy stored in the body will be $\frac{1}{2}Mv^2 + \frac{1}{2}I_m\omega^2$, in which I_m is the moment of inertia of the rotating mass in reference to an axis through the centre, and ω is the angular velocity in reference to the same axis; and the other notation is the same as in the preceding Article. See Article 127. This expression for the energy, in case the bodies are perfectly elastic, will equal the energy lost by the impacting body.)

CHAPTER II.

COMPOSITION AND RESOLUTION OF FORCES.

CONCURRENT FORCES.

39. If two or more forces act upon a material particle, they are said to be *concurrent.* They may all act towards the particle, or from it, or some towards and others from.

40. If several forces act along a material line, they are called *conspiring* forces, and their effect will be the same as if all were applied at the same point.

41. THE RESULTANT of two or more concurrent forces is that force which if substituted for the system will produce the same effect upon a particle as the system.

Therefore, if a force equal in magnitude to the resultant and acting along the same action-line, but in the opposite direction, be applied to the same particle, the system will be in equilibrium.

If the resultant is negative, the equilibrating force will be positive, and vice versa.

Hence, if several concurrent forces are in equilibrium, any one may be considered as equal and opposite to the resultant of all the others.

42. *The resultant of several conspiring forces, equals the algebraic sum of the forces.* That is, if F_1, F_2, F_3, etc., are the forces acting along the same action-line, some of which may be positive and the others negative, and R is the resultant; then

$$R = F_1 + F_2 + F_3 + \text{etc.} = \Sigma F. \qquad (45)$$

43. *If two concurring forces be represented in magnitude and direction by the adjacent sides of a parallelogram, the resultant will be represented in magnitude and direction by the diagonal of the parallelogram.* This is called the *parallelogram of forces.*

If each force act upon a particle for an element of time it will generate a certain velocity. See equation (44). Let

the velocity which F would produce be represented by AB; and that of P by $AD = BC$. These represent the spaces over which the forces respectively would move the particle in a unit of time if each acted separately. If we conceive that the force F moves it from A to B and that the motion is there arrested, and that P is then applied at B, but acting parallel to AD, then will the particle, at the end of two seconds, be at C. If, next, we conceive that each force acts alternately during one-half of a second beginning again at A, the particle will be found at a in one-half of a second; at b at the end of one second; at c at the end of one and one-half seconds; and finally at C at the end of two seconds. If the times be again subdivided the path will be Ad, de, ef, fb, bg, gh, hi, and iC, and it will arrive at C in the same time as before.

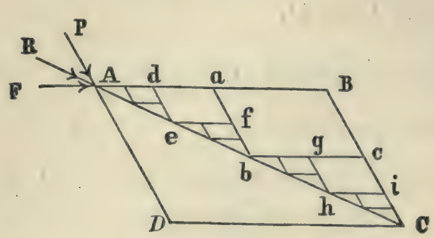

FIG. 19.

As the divisions of the time increase, the number of sides of the polygon increase, each side becoming shorter; and the polygonal path approaches the straight line as a limit. Therefore at the limit, when the force P and F act simultaneously, the particle will move along the diagonal, AC, of the parallelogram. But when they act simultaneously, they will produce their effect in the same time as each when acting separately; and hence, the particle will arrive at C at the end of *one second*. Therefore, a single force R, which is represented by AC, will produce the same effect as P and F, and will be the resultant. If now a force equal and opposite to R act at the same point as the forces F and P, the motion will be arrested and pressure only will be the result. See article 34. Hence, the parallelogram of velocities and of pressures becomes established.*

* This is one of the most important propositions in Mechanics, and has been proved in a variety of ways. One work gives forty-five different proofs. A demonstration given by M. Poisson is one of the most noted of the analytical proofs. Many persons object to admitting the idea of motion in proving the

If θ be the angle between the sides of the parallelogram which represent the forces P and F, and R be the diagonal, or resultant, we have from trigonometry

$$R^2 = F^2 + P^2 + 2PF \cos \theta. \quad (46)$$

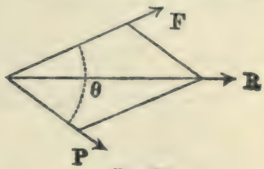

FIG. 20.

If θ exceeds 90 degrees, it must be observed in the solution of problems that $\cos \theta$ will be negative.

If $\theta = 90$ degrees, we have

$$R^2 = F^2 + P^2.$$

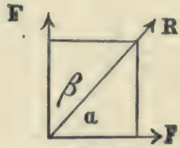

FIG. 21

Also, if $\theta = 90°$, and a be the angle between R and P; and β between R and F; then

$$\left.\begin{array}{l} P = R \cos a; \\ F = R \cos \beta = R \sin a. \end{array}\right\} \quad (47)$$

Squaring and adding, we have

$$P^2 + F^2 = R^2,$$

as before.

The forces P and F are called *component forces.*

44. TRIANGLE OF FORCES. *If three concurrent forces are in equilibrium, they may be represented in magnitude and direction by the sides of a triangle taken in their order; and if the direction of action of one be reversed, it will be the resultant of the other two.*

Thus, in Fig. 19, if AB and BC represent two forces in magnitude and direction, AC will represent the resultant.

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parallelogram of pressures; but we have seen that a pressure when acting upon a free body will produce a certain amount of motion, and that this motion is a measure of the pressure, and hence its use in the proof appears to be admissible. But the strongest proof of the correctness of the proposition is the fact that in all the problems to which it has been applied, the results agree with those of experience and observation.

Since the sines of the angles of a triangle are proportional to the sides opposite, we have

$$\frac{F}{\sin \widehat{P,R}} = \frac{P}{\sin \widehat{F,R}} = \frac{R}{\sin \widehat{F,P}}. \tag{48}$$

POLYGON OF FORCES.

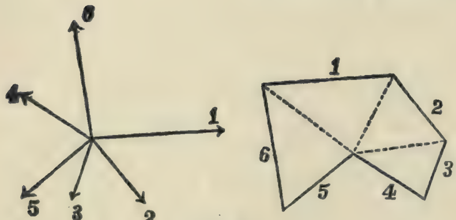


FIG 21a.

45. *If several concurrent forces are represented in magnitude and direction by the sides of a closed polygon taken in their order, they will be in equilibrium.*

This may be proved by finding the resultant of two forces by means of the triangle of forces; then the resultant of that resultant and another force, and so on.

PARALLELOPIPED OF FORCES.

46. *If three concurrent forces not in the same plane are represented in magnitude and direction by the adjacent edges of a parallelopipedon, the resultant will be represented in magnitude and direction by the diagonal; and conversely if the diagonal of a parallelopipedon represents a force, it may be considered as the resultant of three forces represented by the adjacent edges of the parallelopipedon.*

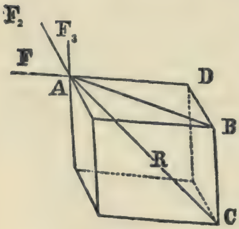


FIG. 22.

In Fig. 22, if AD represents the force F_1 in magnitude and direction, and similarly DB represents F_2, and BC, F_3; then according to the triangle of forces AB will represent the resultant of F_1 and F_2; and AC the resultant of AB and F_3, and hence it represents the resultant of F_1, F_2, and F_3.

If F_1, F_2, and F_3 are at right angles with each other, we have

$$R^2 = F_1^2 + F_2^2 + F_3^2;$$

and if α is the angle $\widehat{R,F_1}$, β of $\widehat{R,F_2}$, and γ of $\widehat{R,F_3}$; then

$$\left.\begin{aligned} F_1 &= R\cos\alpha; \\ F_2 &= R\cos\beta; \\ F_3 &= R\cos\gamma. \end{aligned}\right\} \qquad (49)$$

Squaring these and adding, we have

$$R^2 = F_1^2 + F_2^2 + F_3^2, \text{ as before.}$$

EXAMPLES.

1. When $F = F_1$ and $\theta = 60°$, find R; (See Eq. (46)).
Ans. $R = F\sqrt{3}$.

2. If $F = F_1$ and $\theta = 120°$, find R.

3. If $F = F_1$ and $\theta = 135°$, find R.
Ans. $R = F\sqrt{2 - \sqrt{2}}$.

4. If $F = 2F_1 = 3R$, find θ.

5. If $\frac{3}{5}F = \frac{3}{4}F_1 = R$, find the angle $\widehat{F,F_1}$.
Ans. 90°.

6. If $F = 7$, $F_1 = 9$, and $\theta = 25°$, find R and angle $\widehat{F,R}$.

7. A cord is tied around a pin at a fixed point, and its two ends are drawn in different directions by forces F and P. Find θ when the pressure upon the pin is $R = \frac{1}{2}(P + F)$.

$$Ans.\ \cos\theta = \frac{2PF - 3(P^2 + F^2)}{8PF}.$$

8. When the concurring forces are in equilibrium, prove that

$$P : F : R :: \sin\widehat{F,R} : \sin\widehat{P,R} : \sin\widehat{P,F}.$$

9. If two equal rafters support a weight W at their upper ends, required the compression on each. Let the length of

each rafter be a and the horizontal distance between their lower ends be b.

Ans. $\dfrac{a}{\sqrt{4a^2-b^2}}W.$

10. If a block whose weight is 200 pounds is so situated that it receives a pressure from the wind of 25 pounds in a due easterly direction, and a pressure from water of 100 pounds in a due southerly direction; required the resultant pressure and the angle which the resultant makes with the vertical.

RESOLUTION OF CONCURRENT FORCES.

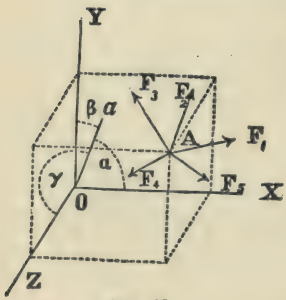


FIG. 23.

47. Let there be many concurrent forces acting upon a single particle, and the whole system be referred to rectangular co-ordinates.

Let F_1, F_2, F_3, etc., be the forces acting upon a particle at A;
x, y, z the co-ordinates of A;
α_1, α_2, etc., the angles which the direction-lines of the respective forces make with the axis of x;
β_1, β_2, etc., the angles which they make with y;
γ_1, γ_2, etc., the angles which they make with z; and
X, Y, and Z, the algebraic sum of the components of the forces when resolved parallel to the axes x, y, and z, respectively.

Then, according to equations (45) and (49), we have for equilibrium;

$$\left.\begin{aligned} X &= F_1\cos\alpha_1 + F_2\cos\alpha_2 + F_3\cos\alpha_3 + \text{etc.} = \Sigma F\cos\alpha = 0; \\ Y &= F_1\cos\beta_1 + F_2\cos\beta_2 + F_3\cos\beta_3 + \text{etc.} = \Sigma F\cos\beta = 0; \\ Z &= F_1\cos\gamma_1 + F_2\cos\gamma_2 + F_3\cos\gamma_3 + \text{etc.} = \Sigma F\cos\gamma = 0; \end{aligned}\right\} \quad (50)$$

If they are not in equilibrium, let R be the resultant, and by introducing a force equal and opposite to the resultant, the system will be in equilibrium.

Let a, b and c be the angles which the resultant makes with the axes x, y and z respectively; then

$$\begin{aligned} X &= R\cos a; \\ Y &= R\cos b; \\ Z &= R\cos c. \end{aligned} \tag{51}$$

Squaring and adding, we have

$$X^2 + Y^2 + Z^2 = R^2 \tag{52}$$

If $R = 0$ equations (51) reduce to (50).

When the forces are in equilibrium any one of the F-forces may be considered as a resultant (reversed) of all the others. Equations (50) are therefore general for concurring forces.

The values of the angles α, β, γ, etc., may be determined by drawing a line *from the origin* parallel to and in *the direction of the action of the force*, and measuring the angles from the axes to the line as in Analytical Geometry. The forces may always be considered as positive, and hence the signs of the terms in (50) will be the same as those of the trigonometrical functions. In Fig. 23 the line Oa is parallel to F_2, and the corresponding angles which it makes with the axes are indicated.

If all the forces are in the plane $x\,y$ then γ_1, γ_2, etc. $= 90°$, and (50) becomes

$$\left.\begin{aligned} X &= \Sigma F\cos\alpha = 0; \\ Y &= \Sigma F\cos\beta = 0. \end{aligned}\right\} \tag{53}$$

CONSTRAINED EQUILIBRIUM.

48. A body is constrained when it is prevented from moving freely under the action of applied forces.

If a particle is constrained to remain at rest on a surface under the action of any number of concurring forces, the resultant of all the applied forces must be in the direction of the normal to the surface at that point.

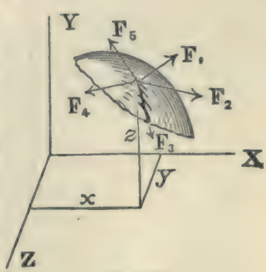


FIG. 24.

For, if the resultant were inclined to the normal, it could be resolved into two components, one of which would be

tangential, and would produce motion; and the other normal, which would be resisted by the surface.

Let $N =$ the normal reaction of the surface, which will be equal and opposite to the resultant of all the impressed forces;

$\theta_x =$ the angle (N,x);

$\theta_y =$ the angle (N,y);

$\theta_z =$ the angle (N,z);

$L = \phi(x, y, z) = 0$, be the functional equation of the surface; and

F_1, F_2, F_3, etc., be the impressed forces.

Then from (51) and (52), we have

$$\left.\begin{aligned} X &= N\cos\theta_x; \\ Y &= N\cos\theta_y; \\ Z &= N\cos\theta_z; \\ X^2 + Y^2 + Z^2 &= N^2. \end{aligned}\right\} \qquad (54)$$

From Calculus we have

$$\cos\theta_x = \frac{1}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2 + \left(\frac{dx}{dz}\right)^2}}$$

$$= \frac{\left(\frac{dL}{dx}\right)}{\sqrt{\left(\frac{dL}{dx}\right)^2 + \left(\frac{dL}{dy}\right)^2 + \left(\frac{dL}{dz}\right)^2}} \qquad (55)$$

and similarly for $\cos\theta_y$ and $\cos\theta_z$.

These values in (54) readily give

$$\frac{X}{\left(\frac{dL}{dx}\right)} = \frac{Y}{\left(\frac{dL}{dy}\right)} = \frac{Z}{\left(\frac{dL}{dz}\right)}. \qquad (56)$$

After substituting the values of $\cos\theta_x$, $\cos\theta_y$, and $\cos\theta_z$ in

(54), multiply the first equation by dx, the second by dy, the third by dz, add the results, and reduce by the equation

$$\left(\frac{dL}{dx}\right) dx + \left(\frac{dL}{dy}\right) dy + \left(\frac{dL}{dz}\right) dz = 0;$$

which is the total differential of the equation $L = 0$; and we have

$$Xdx + Ydy + Zdz = 0. \qquad (57)$$

Equations (56) give two independent simultaneous equations which, combined with the equation of the surface, will determine the point of equilibrium if there be one. Equation (57) is one of condition which will be satisfied if there be equilibrium.

To deduce (55) let $f\,(x', y') = 0$, and $f'\,(x', z') = 0$, be the equations of the normal to the surface at the point where the forces are applied. In Fig. 25 let Oa be drawn through the origin of co-ordinates parallel to the required normal, then will dx', dy' and dz' be directly proportional to the co-ordinates of a;

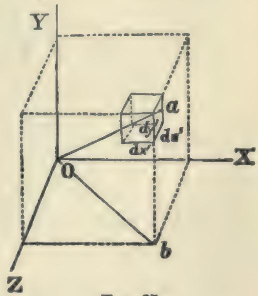


Fig. 25.

$$\therefore \cos aOx = \cos \theta_x = \frac{x}{Oa}$$

$$= \frac{dx'}{\sqrt{dx'^2 + dy'^2 + dz'^2}}$$

$$= \frac{1}{\sqrt{1 + \left(\frac{dy'}{dx'}\right)^2 + \left(\frac{dz'}{dx'}\right)^2}}.$$

But the normal is perpendicular to the tangent plane, and hence the projections of the normal are perpendicular to the traces of the tangent plane. The Equation of Condition of Perpendicularity is of the form $1 + aa' = 0$ (See Analytical Geometry); in which $a' = \frac{dy'}{dx'}$, and $a = \frac{dy}{dx}$; the latter of which is deduced from the equation of the surface;

$$\therefore\ 1 + \frac{dy'}{dx'}\frac{dy}{dx} = 0; \text{ and similarly}$$

$$1 + \frac{dz'}{dx'}\frac{dz}{dx} = 0;$$

hence

$$\frac{dy'}{dx'} = -\frac{dx}{dy} = \frac{\left(\frac{dL}{dy}\right)}{\left(\frac{dL}{dx}\right)}; \text{ and } \frac{dz'}{dx'} = -\frac{dx}{dz} = \frac{\left(\frac{dL}{dz}\right)}{\left(\frac{dL}{dx}\right)};$$

the last terms of which contain the partial differential co-efficients deduced from the equation of the surface. These, substituted in the value of cos θ_x above, and reduced, give equation (55).

CONSTRAINED EQUILIBRIUM IN A PLANE.

49. If all the forces are in the plane of a curve, let the plane yx coincide with that plane; then $Z = 0$ and (56) becomes

$$\left.\begin{array}{l} \dfrac{X}{\left(\frac{dL}{dx}\right)} = \dfrac{Y}{\left(\frac{dL}{dy}\right)}; \\ \text{or, } Xdx = -Ydy; \end{array}\right\} \quad (58)$$

$$\text{and, } Xdx + Ydy = 0; \quad (59)$$

in which the first of (58) may be used when the equation of the curve is given as an implicit function; and the second of (58), or (59), when the equation is an explicit function.

When the particle is not constrained it has three degrees of freedom (equations (50)); when confined to a surface, two degrees (equations (56)); and when confined to a plane curve, only one degree (equation (58)).

EXAMPLES.

1. A body is suspended vertically by a cord which passes over a pulley and is attached to another weight which rests upon a plane; required the position of equilibrium.

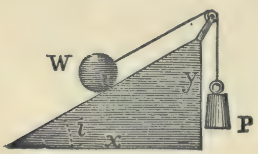


FIG. 26.

In Fig. 26, let the pulley be at the upper end of the plane and the cord and plane perfectly smooth. The weight P is equivalent to a force

which acts parallel to the plane, tending to move the weight W up it.

Let $W =$ the weight on the plane, which acts vertically downwards;

$P =$ the weight suspended by the cord;

$i =$ the inclination of the plane to the horizontal; and

$L = -y + ax + b = 0$, be the equation of the plane.

Then
$$X = P \cos i;$$
$$Y = -W + P \sin i;$$
$$a = \tan i = \frac{\sin i}{\cos i};$$
$$\left(\frac{dL}{dy}\right) = -1, \text{ and } \left(\frac{dL}{dx}\right) = a;$$

and these in (58) give
$$P = W \sin i;$$
which only establishes a relation between the constants, and thus determines the relation which must exist in order that there may be equilibrium; and since the variable co-ordinates do not appear, there will be equilibrium at all points along the plane when $P = W \sin i$.

The equation of the line, given explicitly, is
$$y = ax + b;$$
$$\therefore dy = a\,dx;$$
which in the 2nd of (58), or in (59), gives, $P = W \sin i$ as before.

2. Two weights P and W are fastened to the ends of a cord, which passes over a pulley O; the weight W rests upon a vertical plane curve, and P hangs freely; required the position of equilibrium.

The applied forces at W are the weight W, acting vertically downward; the tension P on the string; and the normal reaction of the curve.

[Consider the weight W and pulley O as reduced to points.]

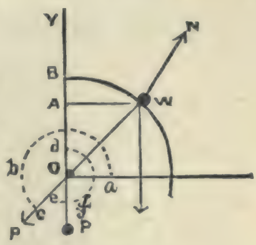


FIG. 27.

Take the origin of co-ordinates at the

pulley, y vertical and positive upwards, and x positive to the right. The angle between $+x$ and P is abc; between $+y$ and P, dfe. Let $AW = x$, $OA = y$, $OW = r$, $WOa = \theta$; then

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r},\ r^2 = x^2 + y^2;$$

$$\begin{aligned} X &= W\cos 270^\circ + P\cos abc = 0 + P\cos(180^\circ + \theta) \\ &= -P\cos\theta, \\ Y &= W\sin 270^\circ + P\sin(180^\circ + \theta) \\ &= -W - P\sin\theta; \end{aligned}$$

and (59) becomes

$$-P\cos\theta\, dx - (W + P\sin\theta)\, dy = 0;$$

or

$$-Wdy = P\frac{xdx + ydy}{r} = Pdr. \qquad (a)$$

3. Let the given curve be a parabola, in which the origin and pulley are at the focus, the axis vertical, and the vertex of the curve above the origin.

The equation of the curve will be

$$x^2 = 2p(-y + \tfrac{1}{2}p),$$

in which $2p$ is the principal parameter. (See *Analyt. Geom.*)

Differentiating gives

$$xdx = -pdy,$$

which substituted in (a) gives

$$-Wdy = P\frac{-p + y}{\pm\sqrt{x^2 + y^2}}dy,$$

or

$$W = \pm P\frac{y - p}{\sqrt{y^2 - 2py + p^2}} = \pm P;$$

which simply establishes a relation between constants; and therefore, if they are in equilibrium at any point they will be at every point, and there will be no equilibrium unless the weights are numerically equal.

4. Let the curve be a circle in which the origin and pulley are at a distance a above the centre of the circle.

Since y is negative downwards, we have for the equation of the circle,

$$(a + y)^2 + x^2 = R^2.$$

Differentiating gives

$$xdx = -(a + y)\,dy,$$

which substituted in (a) gives

$$r = \frac{P}{W}a.$$

5. Let the curve be an hyperbola having the origin and pulley at the centre of the hyperbola, the axis of the curve being vertical.

The equation of the curve will be $b^2y^2 - a^2x^2 = a^2b^2$, and if e be the eccentricity, we find

$$y = \frac{bW}{e(W^2 - e^2P^2)^{\frac{1}{2}}}.$$

6. Required the curve such that the weight W may be in equilibrium with the weight P at all points of the curve.

This relation requires that the relation between y and x in equation (a) shall be true for all assumed values of P and W. We have

$$-Wdy = P\frac{xdx + ydy}{\sqrt{x^2 + y^2}}.$$

Integrating gives

$$-Wy + C = P\sqrt{x^2 + y^2}.$$

Squaring

$$W^2y^2 - 2CWy + C^2 = P^2x^2 + P^2y^2,$$

or

$$P^2x^2 + (P^2 - W^2)\,y^2 + 2CWy - C^2 = 0\,;$$

which is an equation of the second degree, and hence represents a conic.

If $P = W$, it is a parabola.
If $P > W$, it is an ellipse.
If $P < W$, it is a hyperbola.

The origin is at the focus.

7. A particle is placed on the concave surface of a smooth sphere and acted upon by gravity, and also by a repulsive

force, which varies inversely as the square of the distance from the lowest point of the sphere; find the position of equilibrium of the particle.

Take the lowest point of the sphere for the origin of coördinates, y positive upwards, and the equation of the surface will be

$$L = x^2 + y^2 + z^2 - 2Ry = 0.$$

Let r be the distance of the particle from the lowest point; then

$$r^2 = x^2 + y^2 + z^2 = 2Ry. \qquad (b)$$

Let μ be the measure of the repulsive force at a unit's distance; then the forces will be

$$\frac{\mu}{r^2} = \frac{\mu}{2Ry}, \text{ and } mg = w = \text{the weight of the particle.}$$

$$\therefore X = \frac{\mu}{2Ry} \cdot \frac{x}{r}, \quad Y = \frac{\mu}{2Ry} \cdot \frac{y}{r} - w, \quad Z = \frac{\mu}{2Ry} \cdot \frac{z}{r};$$

which in (56) give, after reduction,

$$y = \frac{\mu^{\frac{2}{3}}}{2R^{\frac{1}{3}} w^{\frac{2}{3}}};$$

which in (b) gives, $r^3 = \frac{\mu}{w} R.$

To see if these values satisfy equation (57), substitute in it the values of X, Y, Z, and the final values of y and r, and we find,

$$xdx + ydy - Rdy + zdz = 0;$$

which is the differential of equation (b), and hence is true.

[This is the theory of *the Electroscope.*]

8. A particle on the surface of an ellipsoid is attracted by forces which vary directly as its distance from the principal planes of section; determine the position of equilibrium.

Let

$$L = \phi\,(x, y, z,) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0,$$

be the equation of the surface;

$$\therefore \left(\frac{dL}{dx}\right) = \frac{2x}{a^2}, \quad \left(\frac{dL}{dy}\right) = \frac{2y}{b^2}, \quad \left(\frac{dL}{dz}\right) = \frac{2z}{c^2};$$

and let the x, y, and *z-components* of the forces be respectively,

$$X = -\mu_1 x, \qquad Y = -\mu_2 y, \qquad Z = -\mu_3 z;$$

and (56) will give,

$$\mu_1 a^2 = \mu_2 b^2 = \mu_3 c^2;$$

which simply establishes a relation between the constants; and hence when this relation exists the particle may be at rest at any point on the surface.

The result may be put in the form,

$$\frac{\mu_1}{a^{-2}} = \frac{\mu_2}{b^{-2}} = \frac{\mu_3}{c^{-2}} = \frac{\mu_1 + \mu_2 + \mu_3}{a^{-2} + b^{-2} + c^{-2}}$$

MOMENTS OF FORCES.

50. Def. *The moment of a force in reference to a point is the product arising from multiplying the force by the perpendicular distance of the action-line of the force from the point.*

Thus, in Fig. 28, if O is the point from which the perpendicular is drawn, F the force, and Oa the perpendicular, then the moment of F is

$$F.Oa = Ff;$$

in which f is the perpendicular Oa.

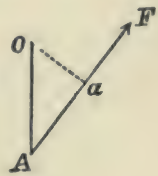


Fig. 28.

51. Nature of a moment. The moment of a force measures the turning or twisting effect of a force. Thus, in Fig. 28, if the particle upon which the force F acts is at A,

and if we conceive that the point O is rigidly connected to A, the force will tend to move the particle about O, and it is evident that this effect varies directly as F. If the action-line of F passed through O it would have no tendency to move the particle about that point, and the greater its distance from that point the greater will be its effect, and it will vary directly as that distance; hence, *the measure of the effect of a moment varies as the product of the force and perpendicular; or as*

$$cFf;$$

where c is a constant. But as c may be chosen arbitrarily, we make it equal to unity, and have simply Ff, as given above.

52. Def. The point O from which the perpendiculars are drawn is chosen arbitrarily, and is called the *origin of moments*. When the system is referred to rectangular coördinates, the *origin of moments* may, or may not, coincide with the origin of coördinates. The solution of many problems is simplified by taking the origin of moments at a particular point.

53. The lever arm, or, simply, *the arm*, of a force is the perpendicular from the origin of moments to the action-line of the force. Thus, in Fig. 29, Oa is the arm of the force F_1; Oc that of the force F_3, etc. *Generally, the arm* is the perpendicular distance of the action-line from the *axis* about which the system is supposed to turn.

54. The sign of a moment is considered *positive* if it tends to turn the system in a direction opposite to that of the hands of a watch; and *negative*, if in the opposite direction. This is arbitrary, and the opposite directions may be chosen with equal propriety; but this agrees with the direction in which the angle is computed in plane trigonometry. *Generally* we shall consider those moments as *positive* which tend to turn the system in the direction indicated by the natural order of the letters, that is, *positive* from $+x$ to $+y$; from $+y$ to $+z$; then from $+z$ to $+x$; and *negative* in the reverse direction.

The value of a moment may be represented by a straight line drawn from the origin and along the line about which

rotation tends to take place, *in one direction for a positive value*, and in the opposite direction for a negative one.

55. THE COMPOSITION AND RESOLUTION of moments may be effected in substantially the same manner as for forces. They may be added, or subtracted, or compounded, so that a resultant moment shall produce the same effect as any number of single moments. The general proof of this proposition is given in the next Chapter.

56. A MOMENT AXIS is a line passing through the origin of moments and perpendicular to the plane of the force and arm.

57. THE MOMENT OF A FORCE REFERRED TO A MOMENT AXIS *is the product of the force into the perpendicular distance of the force from the axis.*

If, in Fig. 29, a line is drawn through O perpendicular to the plane of the force and arm, it will be a moment axis, and the turning effect of F_1 upon that axis will be the same wherever applied, providing that its arm Oa remains constant.

If the force is not perpendicular to the arbitrarily chosen axis, it may be resolved into two forces, one of which will be perpendicular (but need not intersect it) and the other parallel to the axis. The moment of the former component will be the same as that given above, but the latter will have no moment in reference to *that* axis although it may have a moment in reference to another axis perpendicular to the former.

58. THE MOMENT OF A FORCE IN REFERENCE TO A PLANE TO WHICH IT IS PARALLEL *is the product of the force into the distance of its action-line from the plane.*

59. *If any number of concurring forces are in equilibrium the algebraic sum of their moments will be zero.*

Let F_1, F_2, F_3, etc., Fig. 29, be the forces acting upon a particle at A; and O the assumed origin of moments. Join O and A, and let fall the perpendiculars Oa, Ob, Oc, etc., upon the action-lines of the respective forces, and let

$$Oa = f_1;\ Ob = f_2;\ Oc = f_3;\ \text{etc.}$$

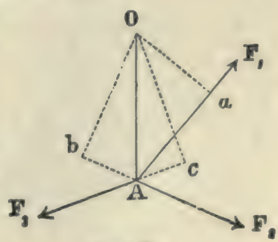


FIG. 29.

Resolve the forces perpendicularly to the line OA; and since they are in equilibrium, the algebraic sum of these components will be zero; hence,

$$F_1 \sin OAF_1 + F_2 \sin OAF_2 + F_3 \sin OAF_3 + \text{etc.} = 0;$$

$$\text{or, } F_1 \frac{Oa}{OA} + F_2 \frac{Ob}{OA} + F_3 \frac{Oc}{OA} + \text{etc.} = 0.$$

Multiply by OA, and we have

$$F_1 Oa + F_2 Ob + F_3 Oc + \text{etc.} = 0;$$

$$\text{or, } F_1 f_1 + F_2 f_2 + F_3 f_3 + \text{etc.} = \Sigma F f = 0. \qquad (60)$$

It is evident that any one of these moments may be taken as the resultant of all the others.

MOMENTS OF CONCURRING FORCES WHEN THE SYSTEM IS REFERRED TO RECTANGULAR AXES.

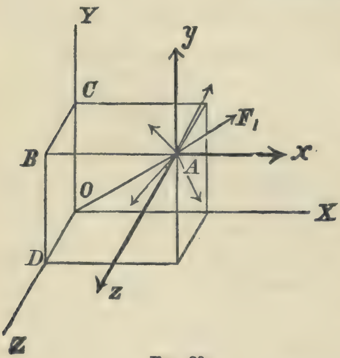


FIG. 30.

60. Let A, Fig. 30, be the point of application of the forces F_1, F_2, F_3, etc., and O the origin of coördinates, and also the origin of moments. Let x, y, and z be the coördinates of the point A. Resolving the forces parallel to the coördinate axes, we have, from equation (50),

$$X = \Sigma F \cos \alpha;$$
$$Y = \Sigma F \cos \beta;$$
$$Z = \Sigma F \cos \gamma.$$

The *X-forces* prolonged will meet the *plane* of yz in B; and will tend to turn the system about the axis of y, in reference to which it has the arm $BC = z$; and also about z, in reference to which it has the arm $BD = y$. Hence, employing the notation already established, we have for the moment of the sum of the components parallel to x,

$$-Xy, \text{ and } +Xz.$$

Similarly for the *y-components* we find the moments,

$$+\,Yx, \text{ and } -\,Yz;$$

and for the *z-components*,

$$-\,Zx, \text{ and } +\,Zy.$$

The moment Xy tends to turn the system one way about the axis of z, and Yx tends to turn it about the same axis, but in the opposite direction; and hence, the combined effect of the two will be their algebraic sum; or

$$Yx - Xy.$$

But since there is equilibrium the sum will be zero. Combining the others in the same manner, we have, for *the moments of concurring forces*, in equilibrium:

$$\left.\begin{array}{lr} \text{In reference to the axis of } x \ldots & Zy - Yz = 0; \\ \text{“ \quad “ \quad “ \quad “ \quad “ \quad “ } y \ldots & Xz - Zx = 0; \\ \text{“ \quad “ \quad “ \quad “ \quad “ \quad “ } z \ldots & Yx - Xy = 0. \end{array}\right\} \quad (61)$$

The third equation may be found by eliminating z from the other two; hence, when X, Y, and Z are known, they are the equations of a straight line; and are *the equations of the resultant.*

If the origin of moments be at some other point, whose coördinates are x', y', and z'; and the coördinates of the point A in reference to the origin of moments be x_1, y_1, and z_1; then will the lever arms be

$$x_1 = x - x'; \; y_1 = y - y'; \text{ and } z_1 = z - z'.$$

When the system is referred to rectangular coördinates the arm of the force, referred to the z-axis, is

$$x \cos \beta - y \cos \alpha,$$

in which y and x are the coördinates of *any point* of the action-line of the force; and α is the angle which the action-line makes with the axis of x, and β the angle which it makes with y.

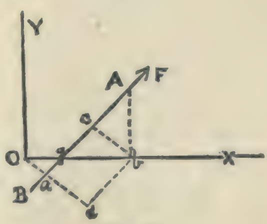


FIG. 81.

In Fig. 31, let BF be the action-line of the force F, O the origin of coördinates, A any point in the line AF, of which the coördinates $x = Ob$, and $y = Ab$. Draw Od and bc perpendicular to AF, and bd from b parallel to BF. The origin of moments being at O, Oa will be the arm of the force.

We have

$$Agb = \alpha = cbA,$$
$$cAb = \beta = bOd,$$
$$cb = y \cos \alpha = ad,$$
$$Od = x \cos \beta;$$
$$\therefore Oa = Od - ad = x \cos \beta - y \cos \alpha. \qquad (61a)$$

If there are three coördinate axes, this will be the arm in reference to the axis of z; and if there be many forces, the sum of their moments in reference to that axis, will be

$$\Sigma F(x \cos \beta - y \cos \alpha).$$

Examples.

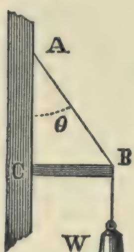


Fig. 32.

1. A weight W is attached to a string, which is secured at A, Fig. 32, and is pushed from a vertical by a strut CB; required the pressure F on BC when the angle CAB is θ.

The forces which concur at B are the weight W, the pressure F, and the tension of the string AB. Take the origin of moments at A, and we have

$$-W.BC + F.AC + \mathit{tension} \times 0 = 0;$$

$$\therefore F = W\frac{BC}{AC} = W \tan \theta.$$

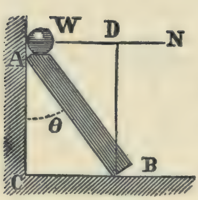


Fig. 33.

2. A brace, AB, rests against a vertical wall and upon a horizontal plane, and supports a weight W at its upper end; required the compression upon the brace and the thrust at A when the angle CAB is θ; the end B being held by a string BC.

The concurring forces at A, are W, acting vertically downward, the reaction of the wall N acting horizontally, and the reaction of the brace F.

Take the origin of moments at B, we have

$$-N.DB + W.CB + F.0 = 0;$$

$$\therefore N = W \tan \theta.$$

Taking the origin of moments at D, we have

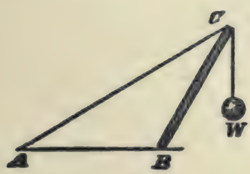
$$W.AD - F.DB \sin \theta + N.0 = 0;$$

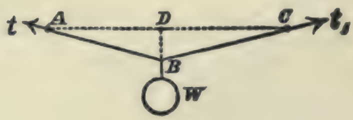
$$\therefore F = W \sec \theta.$$

3. A rod whose length is $BC = l$ is secured at a point B, in a horizontal plane, and the end C is held up by a cord AC so that the angle BAC is θ, and the distance $AB = a$; required the tension on AC and compression on BC, due to a weight W applied at C.

Ans. *Compression* $= \dfrac{l}{a} W \cot \theta.$

4. A cord whose length $ABC = l$ is secured at two points in a horizontal line, and a weight W is suspended from it at B; required the tension on each part of the cord.





CHAPTER III.

PARALLEL FORCES.

61. Bodies are extended masses, and forces may be applied at any or all of their points, and act in all conceivable directions, as in Fig. 34.

FIG. 34.

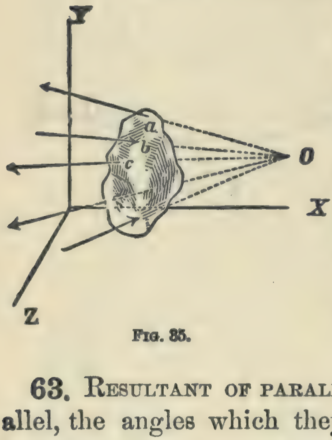


FIG. 35.

62. SUPPOSE THAT THE ACTION-LINES OF ALL THE FORCES ARE PARALLEL TO EACH OTHER. This is a special case of concurrent forces, in which the point of meeting of the action-lines is at an infinite distance. In Fig. 35, let the points a, b, c, etc., which are on the action-lines of the forces and within the body, be the points of application of the forces, and O the point where they would meet if prolonged. If the point O recedes from the body, while the points of application a, b, c, etc. remain fixed, the action-lines of the forces will approach parallelism with each other, and at the limit will be parallel.

63. RESULTANT OF PARALLEL FORCES. The forces being parallel, the angles which they make with the respective axes, including those of the resultant, will be equal to each other Hence,

$$a = a_1 = a_2 = a_3\text{, etc.} = a;$$
$$b = \beta_1 = \beta_2 = \beta_3\text{, etc.} = \beta;$$
$$c = \gamma_1 = \gamma_2 = \gamma_3\text{, etc.} = \gamma;$$

and these, in equations (50) and (51), give

$$\begin{aligned} X &= R \cos a = (F_1 + F_2 + F_3 + \text{etc.}) \cos a. \\ Y &= R \cos \beta = (F_1 + F_2 + F_3 + \text{etc.}) \cos \beta. \\ Z &= R \cos \gamma = (F_1 + F_2 + F_3 + \text{etc.}) \cos \gamma. \end{aligned} \tag{62}$$

From either of these, we have

$$R = F_1 + F_2 + F_3 + \text{etc.} = \Sigma F. \tag{63}$$

Hence, *the resultant of parallel forces equals the algebraic sum of the forces.*

From (62), we have

$$R = \sqrt{X^2 + Y^2 + Z^2},$$

which is the same as (52).

MOMENTS OF PARALLEL FORCES.

64. Let F_1, F_2, F_3, etc., be the forces, and x_1, y_1, z_1; x_2, y_2, z_2, etc., be the coördinates of the points of application of the forces respectively (which, as before stated, may be at any point on their action-lines). Then the moments of F_1 will be, according to Article (60),

in reference to the axis of x, $F_1 \cos \gamma \, . \, y_1 - F_1 \cos \beta \, . \, z_1$;
" " " " " " y, $F_1 \cos a \, . \, z_1 - F_1 \cos \gamma \, . \, x_1$;
" " " " " " z, $F_1 \cos \beta \, . \, x_1 - F_1 \cos a \, . \, y_1$;

and similarly for all the other forces. Hence, the *sum of the moments* in reference to the respective axes for equilibrium is,

$$\left.\begin{aligned} (F_1y_1 + F_2y_2 + F_3y_3 + \text{etc.}) \cos \gamma \\ -(F_1z_1 + F_2z_2 + F_3z_3 + \text{etc.}) \cos \beta \end{aligned}\right\} = 0;$$

$$\left.\begin{aligned} (F_1z_1 + F_2z_2 + F_3z_3 + \text{etc.}) \cos a \\ -(F_1x_1 + F_2x_2 + F_3x_3 + \text{etc.}) \cos \gamma \end{aligned}\right\} = 0;$$

$$\left.\begin{aligned} (F_1x_1 + F_2x_2 + F_3x_2 + \text{etc.}) \cos \beta \\ -(F_1y_1 + F_2y_2 + F_3y_3 + \text{etc.}) \cos a \end{aligned}\right\} = 0.$$

These equations will be true for all values of α, β, and γ, if the coefficients of $\cos\alpha$, $\cos\beta$, $\cos\gamma$, are respectively equal to zero; for which case we have

$$\left.\begin{aligned} F_1x_1 + F_2x_2 + F_3x_3 + \text{etc.} &= \Sigma Fx = 0; \\ F_1y_1 + F_2y_2 + F_3y_3 + \text{etc.} &= \Sigma Fy = 0; \\ F_1z_1 + F_2z_2 + F_3z_3 + \text{etc.} &= \Sigma Fz = 0; \end{aligned}\right\} \tag{64}$$

from which the coördinates of the point of application of any one of the forces, as F_1, for instance, may be found so as to satisfy these equations, when all the other quantities are given.

Let the given forces have a resultant. Let a force, as F_1, equation (64), equal and opposite to the resultant, be introduced into the system, then will there be equilibrium. Let ΣFx, ΣFy, ΣFz, include the sum of the respective products for all the forces *except* that of the resultant; R be the resultant, and $\bar{x}$, $\bar{y}$, $\bar{z}$, the coördinates of the point of application of the resultant; so chosen as to satisfy equations (64), then we have

$$R\bar{x} - \Sigma Fx = 0; \quad R\bar{y} - \Sigma Fy = 0; \quad R\bar{z} - \Sigma Fz = 0. \tag{65}$$

Substitute the value of $R = \Sigma F$, in these equations, and we find

$$\bar{x} = \frac{\Sigma Fx}{\Sigma F}; \qquad \bar{y} = \frac{\Sigma Fy}{\Sigma F}; \qquad \bar{z} = \frac{\Sigma Fz}{\Sigma F}; \tag{66}$$

by which the point of application of the resultant becomes known, and, being independent of α, β, and γ, *is a point through which the resultant constantly passes, as the forces are turned about their points of application, the forces constantly retaining their parallelism.* This point is called *the centre of parallel forces.*

65. If the system consists of three forces only, and are in the plane xy, we have

$$\left.\begin{aligned} R &= F_1 + F_2; \\ \bar{x} = \frac{F_1x_1 + F_2x_2}{F_1 + F_2}; &\text{ and } \bar{y} = \frac{F_1y_1 + F_2y_2}{F_1 + F_2}. \end{aligned}\right\} \tag{67}$$

1st. *Consider F_1 and F_2 as positive.* The resultant will equal the arithmetical sum of the forces. Take the origin at a Fig. 36, where the resultant cuts the axis of x; then $\bar{x} = 0$, and the second of (67) gives

$$F_1x_1 = -F_2x_2;$$

and hence, if $F_1 > F_2$, x_2 will exceed x_1; that is, the resultant is nearer the greater force.

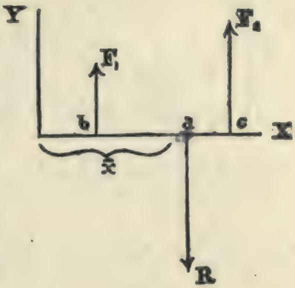


FIG. 36.

2d. *Consider F_2 as negative.*

In this case the resultant equals the difference of the forces. Take the origin at a, Fig. 37, and we have

$$F_1x_1 = F_2x_2;$$

and hence both forces are either at the right or left of the resultant.

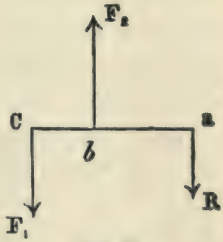


FIG. 37.

3d. Let $F_1 = F_2 = F$, and one of the forces be negative, then

$$R = F - F = 0; \; \bar{x} = \frac{F(x_1 \pm x_2)}{F - F} = \infty \; ; \text{ and } \bar{y} = \infty \; ; \quad (68)$$

that is, the resultant is zero, while the forces may have a finite moment equal to $F(x_1 \pm x_2)$. Such systems are called

COUPLES.

66. *A couple consists of two equal parallel forces acting in opposite directions at a finite distance from each other.*

A statical couple cannot be equilibrated by a single force. It does not produce translation, but simply rotation. *A couple can be equilibrated only by an equivalent couple.*

Equivalent couples are such as have equal moments.

The resultant of several couples is a single couple which will produce the same effect as the component couples.

67. *The arm of a couple is the perpendicular distance between the action-lines of the forces.* Thus, in Fig. 38, let O be the origin of coördinates, and the axis of x perpendicular to the action-line of F; then will the moment of one force be Fx_1, and of the other Fx_2, and hence the *resultant moment will be*

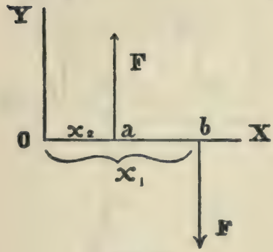


FIG. 38.

$$F(x_1 - x_2) = F.ab; \qquad (69)$$

hence, ab is the arm. If the origin of coördinates were between the forces the moments would be $F(x_1 + x_2) = F.ab$ as before. If the origin be at a we have $F.0 + F.ab = F.ab$ as before.

68. THE AXIS OF A STATICAL COUPLE *is any line perpendicular to the plane of the couple.* The length of the axis may be made proportional to the moment of the couple, and placed on one side of the plane when the moment is positive, and on the opposite side when it is negative; and thus completely represent the couple in magnitude and direction.

If couples are in parallel planes, their axes may be so taken that they will conspire, and hence the resultant couple equals the algebraic sum of all the couples.

If the planes of the couples intersect, their axes may intersect.

Let $O = F.ab =$ the moment of one couple;
$O_1 = F_1.a_1b_1 =$ the moment of another couple;
$\theta =$ the angle between their axes; and
$O_R =$ the resultant of the two couples;

then

$$O_R = \sqrt{O^2 + O_1^2 + 2OO_1 \cos\theta};$$

and this resultant may be combined with another and so on until the final resultant is obtained.

EXAMPLES.

1. Three forces represented in magnitude, direction and *position*, by the sides of a triangle, taken in their order, produce a couple.

2. If three forces are represented in magnitude and position by the sides of a triangle, but whose directions do not follow the *order* of the sides; show that they will have a single resultant.

3. On a straight rod are suspended several weights; $F_1 = 5$ lbs., $F_2 = 15$ lbs., $F_3 = 7$ lbs., $F_4 = 6$ lbs., $F_5 = 9$ lbs., at distances $AB = 3$ ft., $BD = 6$ ft., $DE = 5$ ft., and $EF = 4$ ft.; required the distance AC at which a fulcrum must be placed so that the weights will balance on it; also required the pressure upon it.

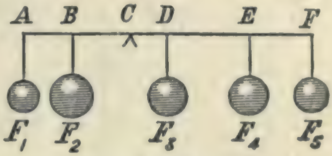


Fig. 39.

4. The whole length of the beam of a false balance is 2 feet 6 inches. A body placed in one scale balances 6 lbs. in the other, but when placed in the other scale it balances 8 lbs.; required the true weight of the body, and the lengths of the arms of the balance.

5. A triangle in the horizontal plane x, y has weights at the several angles which are proportional respectively to the opposite sides of the triangle; required the coördinates of the centre of the forces.

Let x_1, y_1 be the coördinates of A,
x_2, y_2 of B; x_3, y_3 of C;
$\bar{x}, \bar{y}$ of the point of application of the resultant;
then we have

$$(a + b + c)\,\bar{x} = ax_1 + bx_2 + cx_3; \text{ and}$$
$$(a + b + c)\,\bar{y} = ay_1 + by_2 + cy_3.$$

6. If weights in the proportion of 1, 2, 3, 4, 5, 6, 7 and 8 are suspended from the respective angles of a parallelopiped; required the point of application of the resultant.

7. Several couples in a plane, whose forces are parallel, are applied to a rigid right line, as in Fig. 40; required the resultant couple.

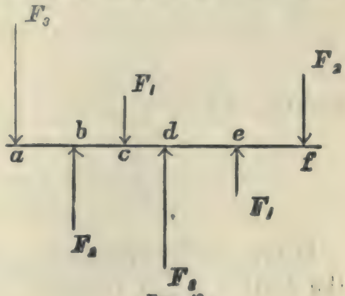


Fig. 40.

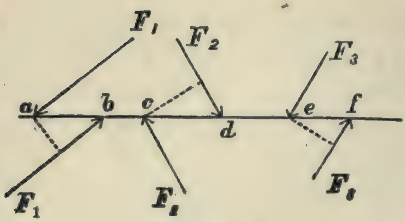


FIG. 41.

8. Several couples in a plane, whose respective arms are not parallel, as in Fig. 41, act upon a rigid right line; required the resultant couple.

CENTRE OF GRAVITY OF BODIES.

69. The action-lines of the force of gravity are normal to the surface of the earth, but, for those bodies which we shall here consider, their convergence will be so small, that we may consider them as parallel. We may also consider the force as the same at all points of the body.

The *centre of gravity of a body* is the point of application of the resultant of the force of gravity as it acts upon every particle of the body. It is the centre of parallel forces. If this point be supported the body will be supported, and if the body be turned about this point it will remain constantly in the centre of the parallel forces.

Let $M =$ the mass of a body;
$m =$ the mass of an infinitesimal element;
$V =$ the volume of the body;
$D =$ the density at the point whose coördinates are x, y, and z;
$R = W =$ the resultant of gravity, which is the weight; and
$\bar{x}$, $\bar{y}$, and $\bar{z}$ be the coördinates of the centre of gravity.

We have, according to equations (63) and (20),

$$R = W = \Sigma gm = M \times g;$$

and (65) becomes

$$\left.\begin{array}{l} \bar{x}\,\Sigma gm = \Sigma gmx;\ \text{or } M\bar{x} = \Sigma mx; \\ \bar{y}\,\Sigma gm = \Sigma gmy;\ \text{or } M\bar{y} = \Sigma my; \\ \bar{z}\,\Sigma gm = \Sigma gmz;\ \text{or } M\bar{z} = \Sigma mz. \end{array}\right\} \qquad (70)$$

If the density is a continuous function of the coördinates of the body we may integrate the preceding expressions The

complete solution will sometimes require two or three integrations, depending upon the character of the problem; but, using only one integral sign, (22) and (70) become

$$M = \int D\, dV;$$

$$\left.\begin{aligned} \bar{x}\int D\, dV &= \int Dx\, dV; \\ \bar{y}\int D\, dV &= \int Dy\, dV; \\ \bar{z}\int D\, dV &= \int Dz\, dV. \end{aligned}\right\} \qquad (71)$$

If the origin of coördinates be at the centre of gravity, then

$$\bar{x} = 0; \ \bar{y} = 0; \ \bar{z} = 0;$$

and hence,

$$\Sigma mx = \int Dx\, dV = 0; \qquad (71a)$$

and similarly for the other values.

If D be constant, this becomes

$$\int_{\prime\prime}^{\prime\prime} x\, dV = 0; \qquad (71b)$$

the limits of integration including the whole body.

If the mass is homogeneous, the density is uniform, and D being cancelled in the preceding equations, we have

$$\left.\begin{aligned} \bar{x} &= \frac{\int x\, dV}{V}; \\ \bar{y} &= \frac{\int y\, dV}{V}; \\ \bar{z} &= \frac{\int z\, dV}{V}. \end{aligned}\right\} \qquad (72)$$

Many solutions may be simplified by observing the following principles:

1. *If the body has an axis of symmetry the centre of gravity will be on that axis.*

2. *If the body has a plane of symmetry the centre of gravity will be in that plane.*

3. *If the body has two or more axes of symmetry the centre of gravity will be at their intersection.*

Hence, the centre of gravity of a physical straight line of uniform density will be at the middle of its length; that of the circumference of a circle at the centre of the circle; that of the circumference of an ellipse at the centre of the ellipse; of the area of a circle, of the area of an ellipse, of a regular polygon, at the geometrical centre of the figures. Similarly the centre of gravity of a triangle will be in the line joining the vertex with the centre of gravity of the base; of a pyramid or cone in the line joining the apex with the centre of gravity of the base.

There is a certain inconsistency in speaking of the centre of gravity of geometrical lines, surfaces, and volumes; and when they are used, it should be understood that a *line* is a *physical* or *material line* whose section may be infinitesimal; a *surface* is a *material section*, or thin plate, or thin shell; and a *volume* is a mass, however attenuated it may be.

When a body has an axis of symmetry, the axis of x may be made to coincide with it, and only the first of the preceding equations will be necessary. If it has a plane of symmetry, the plane $x\,y$ may be made to coincide with it, and only the first and second will be necessary.

70. *Centre of gravity of material lines.*

Let $k =$ the transverse section of the line, and
$ds =$ an element of the length,

then

$$dV = kds;$$

and (71) becomes

$$\left.\begin{aligned}\bar{x}\int Dkds &= \int Dkxds;\\ \bar{y}\int Dkds &= \int Dkyds;\\ \bar{z}\int Dkds &= \int Dkzds.\end{aligned}\right\} \qquad (73)$$

If the transverse section and the density are uniform, we have

$$\left.\begin{aligned} \bar{x} &= \frac{\int x ds}{s}; \\ \bar{y} &= \frac{\int y ds}{s}; \\ \bar{z} &= \frac{\int z ds}{s}. \end{aligned}\right\} \qquad (74)$$

The centre of gravity will sometimes be outside of the line or body, and hence, if it is to be supported at that point, we must conceive it to be rigidly connected with the body by lines which are without weight.

EXAMPLES.

1. Find the centre of gravity of a straight fine wire of uniform section in which the density varies directly as the distance from one end.

Let the axis of x coincide with the line, and the origin be taken at the end where the density is zero. Let δ be the density at the point where $x = 1$; then for any other point it will be $D = \delta x$; and substituting in the first of (73), also making $ds = dx$, we have

$$\bar{x}\int_0^a \delta x dx = \int_0^a \delta x^2 dx; \qquad \therefore \bar{x} = \tfrac{2}{3}a.$$

This corresponds with the distance of the centre of gravity of a triangle from the vertex.

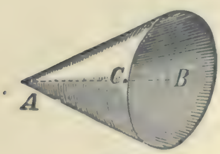


FIG. 42.

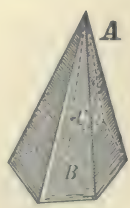


FIG. 43.

2. Find the centre of gravity of a cone or pyramid, whether

right or oblique, and whether the base be regular or irregular.

Draw a line from the apex to the centre of gravity of the base, and conceive that all sections parallel to the base are reduced to this central line. The problem is then reduced to finding the centre of gravity of a physical line in which the density increases as the square of the distance from one end.

Ans. $\bar{x} = \frac{3}{4} a$.

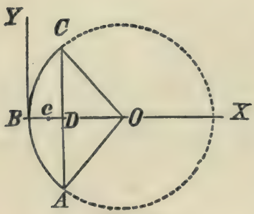


FIG. 44.

3. To find the centre of gravity of a circular arc.

Let the axis of x pass through the centre of the arc, B, and the centre of the circle O; then $\overline{Y} = 0$. Take the origin at B:

and let

$$x = BD,$$
$$y = DC,$$
$$2s = \text{the arc } ABC, \text{ and}$$
$$r = OC = \text{the radius of the circle};$$

then,

$$y^2 = 2rx - x^2.$$

Differentiate, and we have

$$ydy = rdx - xdx;$$

$$\therefore \frac{dy}{r-x} = \frac{dx}{y} \text{ which } = \frac{ds}{r};$$

hence, the first of (74) gives

$$\bar{x} = \frac{r\int_0^y \frac{xdx}{\sqrt{2rx - x^2}}}{s} = \frac{r}{s}\Big[-\sqrt{2rx - x^2} + s\Big]_0^y = r - \frac{ry}{s};$$

which is the distance Bc. Then $cO = r - \left(r - \frac{ry}{s}\right) = \frac{ry}{s}$,

hence, the distance of the centre of gravity of an arc from the

centre of the circle is a fourth proportional to the arc, the radius, and the chord of the arc.

71. *Centre of gravity of plane surfaces.*

Let the coördinate plane xy coincide with the surface; then $dV = dxdy$; $\therefore\ V = \iint dxdy = \int ydx$ or $\int xdy$; and (71) becomes

$$\left.\begin{aligned}\bar{x}\iint D\,dxdy &= \iint D\,xdxdy;\\ \bar{y}\iint D\,dxdy &= \iint D\,ydxdy.\end{aligned}\right\} \qquad (75)$$

The integrals are definite, including the whole area. The order of integration is immaterial, but after the first integration the limits must be determined from the conditions of the problem. If D is constant and the integral is made in respect to y, we have

$$\left.\begin{aligned}\bar{x} &= \frac{\int yxdx}{\int ydx};\\ \bar{y} &= \frac{\frac{1}{2}\int y^2dx}{\int ydx};\end{aligned}\right\} \qquad (76)$$

and if x be an axis of symmetry, the first of these equations will be sufficient.

If the surface is referred to polar coördinates, then $dV = \rho d\rho d\theta$, and $x = \rho\cos\theta$, $y = \rho\sin\theta$, and (71) becomes

$$\left.\begin{aligned}\bar{x} &= \frac{\iint D\rho^2\cos\theta d\rho d\theta}{\iint D\rho d\rho d\theta};\\ \bar{y} &= \frac{\iint D\rho^2\sin\theta d\rho d\theta}{\iint D\rho d\rho d\theta}.\end{aligned}\right\} \qquad (77)$$

EXAMPLES.

1. Find the centre of gravity of a semi-parabola whose equation is $y^2 = 2px$.

Equations (76) become

$$\bar{x} = \frac{\int_0^x \sqrt{2p}\,x^{\frac{3}{2}}\,dx}{\frac{2}{3}xy} = \tfrac{3}{5}x;$$

$$\bar{y} = \frac{\frac{1}{2}\int_0^y 2pxdx}{\frac{2}{3}xy} = \tfrac{3}{8}y.$$

2. Find the centre of gravity of a quadrant of a circle in which the density increases directly as the distance from the centre.

Let $\delta =$ the density at a unit's distance from the centre; then

$$D = \delta\rho \text{ at a distance } \rho;$$

and (77) becomes

$$\bar{x} = \frac{\delta\int_0^{\frac{1}{2}\pi}\int_0^r \rho^3 \cos\theta d\rho d\theta}{\delta\int_0^{\frac{1}{2}\pi}\int_0^r \rho^2 d\rho d\theta} = \tfrac{3}{2}\frac{r}{\pi} = \bar{y}.$$

72. *Centre of gravity of curved surfaces.*

We have for an element of the area
$dV = dx\, dy \times$ *sec. of the angle between the tangent plane and the coördinate plane xy ;* or,

$$dV = \sec.\,\theta \times dx\, dy;$$

$$\therefore V = \iint \sqrt{1 + \left(\frac{dz}{dy}\right)^2 + \left(\frac{dz}{dx}\right)^2}\, dx\, dy;$$

or, in terms of partial differential coefficients

$$V = \iint \frac{\sqrt{\left(\frac{dL}{dx}\right)^2 + \left(\frac{dL}{dy}\right)^2 + \left(\frac{dL}{dz}\right)^2}}{\left(\frac{dL}{dz}\right)} dx\, dy;$$

and, for a homogeneous surface, (72) becomes

$$\left.\begin{aligned} \bar{x} &= \frac{\iint x dV}{V}; \\ \bar{y} &= \frac{\iint y dV}{V}; \\ \bar{z} &= \frac{\iint z dV}{V}; \end{aligned}\right\} \qquad (78)$$

or, the surface may be referred to polar coördinates.

If the surface is one of revolution, let x coincide with the axis of revolution, then

$$\bar{x}\int \pi y ds = \int \pi y x ds. \qquad (78a)$$

EXAMPLE.

Find the centre of gravity of one-eighth of the surface of a sphere contained within three principal planes.

Let the equation of the sphere be

$$L = x^2 + y^2 + z^2 - R^2 = 0;$$

then

$$\frac{dL}{dx} = 2x; \quad \frac{dL}{dy} = 2y; \quad \frac{dL}{dz} = 2z;$$

and the first of (78) becomes

$$\bar{x} = \frac{\iint x \frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2z}}{\frac{1}{2}\pi R^2} dx\, dy$$

$$= \frac{2}{\pi R}\iint \frac{x}{z}\, dx\, dy$$

$$= \frac{2}{\pi R}\iint \frac{x\, dx\, dy}{\sqrt{R^2 - y^2 - x^2}}$$

$$= -\frac{2}{\pi R}\int \sqrt{(R^2 - y^2 - x^2)}dy\Big]_{x=0}^{x=\sqrt{R^2-y^2}}$$

$$= \frac{2}{\pi R}\int \sqrt{(R^2 - y^2)}\, dy$$

$$= \frac{1}{\pi R}\Big[y\sqrt{R^2 - y^2} + R^2 \sin^{-1}\frac{y}{R}\Big]_0^R$$

$$= \tfrac{1}{2}R.$$

Similarly, $\bar{y} = \frac{1}{2}R = \bar{z}$.

This problem may be easily solved by the aid of elementary geometry. Conceive that the surface is divided into an indefinite number of small zones by equidistant planes which are perpendicular to the axis of x, in which case

the area of the zones will be equal to each other. Conceive that these zones are reduced to the axis of x; they will then be uniformly distributed along that axis, and hence the centre of gravity will be $\frac{1}{2}R$ from the centre; and as the surface is symmetrical in respect to each of the three axes, we get the same result in respect to each.

73. *Centre of gravity of volumes, or heavy bodies.*

We have

$$dV = dx\,dy\,dz;$$

$$\therefore\ \bar{x}\iiint D\,dx\,dy\,dz = \iiint Dx\,dx\,dy\,dz; \qquad (79)$$

and similarly for $\bar{y}$ and $\bar{z}$.

If x is an axis of symmetry, (79) is sufficient.

If the surface is referred to polar coördinates, let

$$\phi = AOx,$$
$$\theta = dOA,$$
$$\rho = Og,$$

then,

$$gd = d\rho,$$
$$gf = \rho d\theta,$$
$$gh = \rho\cos\theta d\phi,$$

and,

$$dV = \rho^2 d\rho\cos\theta d\theta d\phi,$$

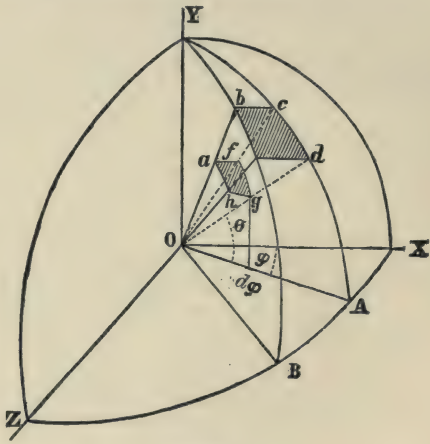


FIG. 45.

also,

$$x = \rho\cos\theta\cos\phi;\ y = \rho\sin\theta;\ \text{and}\ z = \rho\cos\theta\sin\phi.$$

Hence, for a homogeneous body, we have

$$\left.\begin{aligned} V\bar{x} &= \iiint \rho^3\cos^2\theta\cos\phi\,d\rho\,d\theta\,d\phi; \\ V\bar{y} &= \iiint \rho^3\cos\theta\ \sin\theta\,d\rho\,d\theta\,d\phi; \\ V\bar{z} &= \iiint \rho^3\cos^2\theta\sin\phi\,d\rho\,d\theta\,d\phi. \end{aligned}\right\} \qquad (80)$$

If the volume be one of revolution about the axis of x, we have

$$\left.\begin{aligned} dV &= \pi\, y^2\, dx; \\ V\bar{x} &= \pi \textstyle\int y^2\, x\, dx. \end{aligned}\right\} \tag{81}$$

EXAMPLE.

Find the centre of gravity of one-eighth of the volume of a homogeneous ellipsoid, contained within the three *principal* planes.

Let the equation of the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0;$$

then, equation (79) gives,

$$\bar{x} \int_0^c \int_0^Y \int_0^X dx\, dy\, dz = \int_0^c \int_0^Y \int_0^X x\, dx\, dy\, dz.$$

Performing the integration, we have

$$\tfrac{1}{6}\, \pi abc\, \bar{x} = \tfrac{1}{16}\, \pi a^2 bc;$$

$$\therefore\ \bar{x} = \tfrac{3}{8}\, a.$$

Similarly, $\bar{y} = \frac{3}{8}b$, and $\bar{z} = \frac{3}{8}c$.

Performing the above integration in the order of the letters x, y and z, and using the limits in the reverse order as indicated, we have for the *x-limits*.

$$x = 0, \text{ and } x = a\sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}} = X,$$

and the corresponding limits for y will be

$$y = b\sqrt{1 - \frac{z^2}{c^2}} = Y; \text{ and } y = 0.$$

For the *first* member of the equation, we have

$$\int_0^c \int_0^Y \int_0^X dx\, dy\, dz = \int_0^c \int_0^Y x\, dy\, dz = \int_0^c \int_0^Y a\sqrt{1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\, dy\, dz.$$

Consider $\sqrt{1 - \frac{z^2}{c^2}} = B$, a constant in reference to the *y-integration* and we have

$$\int_0^c \tfrac{1}{2}a\left[y\sqrt{B^2-\frac{y^2}{b^2}}+bB^2\sin^{-1}\frac{y}{bB}\right]_0^{\frac{b}{c}\sqrt{c^2-z^2}} dz$$

$$= \tfrac{1}{2}\pi ab\int_0^c\left(1-\frac{z^2}{c^2}\right)dz = \tfrac{1}{3}\pi abc.$$

For the *second* member, we have

$$\int_0^c\int_0^Y\int_0^X x\,dx\,dy\,dz = \int_0^c\int_0^Y \tfrac{1}{2}x^2\,dy\,dz\Big]_0^X = \tfrac{1}{2}a^2\int_0^c\int_0^Y\left(1-\frac{y^2}{b^2}-\frac{z^2}{c^2}\right)dy\,dz$$

Consider z as constant in performing the *y-integration*, and we have

$$\tfrac{1}{2}a^2\int_0^c\left(y-\frac{y^3}{3b^2}-\frac{z^2y}{c^2}\right)dz\Big]_0^Y$$

$$= \tfrac{1}{2}a^2\int_0^c\left[\frac{b}{c}(c^2-z^2)^{\frac{1}{2}} - \frac{b}{3c^3}(c^2-z^2)^{\frac{3}{2}} - \frac{b}{c^3}z^2(c^2-z^2)^{\frac{1}{2}}\right]dz$$

$$= \tfrac{1}{16}\pi a^2bc$$

$$\therefore\ \bar{x} = \frac{\frac{1}{16}\pi a^2bc}{\frac{1}{3}\pi abc} = \tfrac{3}{8}a;$$

as given above.

74. When the centre of gravity of a body is known and the centre of gravity of a part is also known, the centre of the remaining part may be found as follows:—

Let $W =$ the weight of the whole body;
$\bar{x} =$ the distance from the origin to the centre of gravity of the body;
$w_1 =$ the weight of one part;
$x_1 =$ the distance of the centre of w_1 from the same origin;
$w_2 =$ the weight of the other part; and
$x_2 =$ the distance of the centre of w_2 from the same origin;

then

$$w_1x_1 + w_2x_2 = (w_1+w_2)\bar{x} = W\bar{x};$$

and hence,

$$x_2 = \frac{W\bar{x} - w_1x_1}{w_2}. \tag{32}$$

If the body is homogeneous, the volumes may be substituted for the weight.

EXAMPLE.

Let ABC be a cone in which the line BE joins the vertex and the centre of gravity of the base; and the cone ADC, having its apex D, on the line BE, and the same base AC be taken from the former cone, required the centre of gravity of the remaining part, $ADCB$.

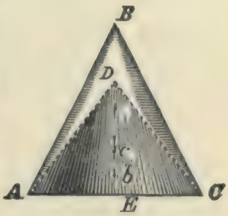


FIG. 46.

Let $V =$ the volume of ACB,

$a = BE$,

$a_1 = DE$,

$\bar{x} = Ec = \frac{1}{4}a =$ the distance of the centre of ABC from E,

$x_1 = Eb = \frac{1}{4}a_1 =$ the distance of the centre of ADC from E;

then,

$$\frac{a_1}{a} V = \text{the volume of } ACD,$$

and (82) becomes

$$x_2 = \frac{V.\frac{1}{4}a - V.\frac{a_1}{a}.\frac{1}{4}a_1}{V - V.\frac{a_1}{a}}$$

$$= \frac{1}{4}(a + a_1).$$

CENTROBARIC METHOD, OR

75. THEOREMS OF PAPPUS OR OF GULDINUS.

Multiply both members of the second of (74) by 2π, and it may be reduced to

$$2\pi\bar{y}s = 2\pi \int y ds, \tag{83}$$

the second member of which is the area generated by the revolution of a line whose length is s about the axis of x, and the

first member is the circumference described by the centre of gravity of the line, multiplied by the length of the line; hence, *the area generated by the revolution of a line about a fixed axis equals the length of the line multiplied by the circumference described by the centre of gravity of the line.*

This is one of the theorems, and the following is the other.

From the second of (76), we find

$$2\pi\bar{y}\,A = \int \pi y^2 dx.$$

The right-hand member, integrated between the proper limits, is the volume generated by the revolution of a plane area about the axis of x. The plane area must lie wholly on one side of the axis. In the first member of the equation, A is the area of the plane curve, and $2\pi\bar{y}$ is the circumference described by its centre of gravity. Hence, *the volume generated by the revolution of a plane curve which lies wholly on one side of the axis, equals the area of the curve multiplied by the circumference described by its centre of gravity.*

Examples.

1. Find the surface of a ring generated by the revolution of a circle, whose radius is r, about an axis whose distance from the centre is c.

Ans. $4\pi^2 rc$.

2. The surface of a sphere is $4\pi r^2$, and the length of a semicircumference is πr; required the ordinate to the centre of gravity of the arc of a semicircle.

3. Required the volume generated by an ellipse, whose semi-axes are a and b, about an axis of revolution whose distance from the centre is c; c being greater than a or b.

(Observe that the volume will be the same for all positions of the axes a and b in reference to the axis of revolution.)

4. The volume of a sphere is $\frac{4}{3}\pi r^3$, and the area of a semicircle is $\frac{1}{2}\pi r^2$; show that the ordinate to the centre of gravity of the semicircle is $\bar{y} = \dfrac{4r}{3\pi}$.

76. ADDITIONAL EXAMPLES.

1. Find the centre of gravity of the quadrant of the circumference of a circle contained between the axes x and y, the origin being at the centre.

$$Ans.\ \bar{x} = \frac{2r}{\pi} = \bar{y}.$$

2. Find the distance of the centre of gravity of the arc of a cycloid from the vertex, r being the radius of the generating circle.

$$Ans.\ \bar{y} = \tfrac{2}{3}r.$$

3. Find the centre of gravity of one-half of the loop of a leminiscate, of which the equation is $r^2 = a^2 \cos 2\theta$, l being the length of the half loop.

$$Ans.\ \bar{x} = \frac{a^2}{\sqrt{2}l};\ \bar{y} = \frac{a^2}{l}\,\frac{\sqrt{2}-1}{\sqrt{2}}.$$

4. Find the centre of gravity of the helix whose equations are

$$x = a\cos\phi,\ y = a\sin\phi,\ z = na\,\phi,$$

the helix starting on the axis of x.

$$Ans.\ \bar{x} = na\,\frac{y}{z};\ \bar{y} = na\,\frac{a-x}{z};\ \text{and } \bar{z} = \tfrac{1}{2}z.$$

5. Find the centre of gravity of the perimeter of a triangle in space.

6. If x_0 and y_0 are initial points of a curve, find the curve such that $m\bar{x} = x - x_0$, and $n\bar{y} = y - y_0$.

7. A curve of given length joins two fixed points; required its form so that its centre of gravity shall be the lowest possible.

(This may be solved by the Calculus of Variations).

Ans. A Catenary.

8. Find the centre of gravity of a trapezoid.

Let $ADEB$ be the trapezoid, in which DE and AB are the parallel sides. Produce AD and BE until they meet in C, and join C with F, the middle point of the base; then the centre of gravity will be at some point g on this line. The centre of the triangle ACB will be on CF, and at a distance of $\frac{1}{3}CF$ from F; and similarly that of DCE will be on the same line, and at a distance of $\frac{1}{3}CG$ from G; then, by (82) we may find Fg.

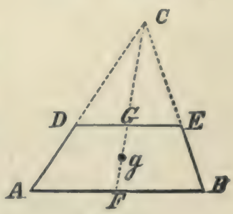


FIG. 47.

$$Ans.\ Fg = \tfrac{1}{3}FG\,\frac{AB + 2DE}{AB + DE}.$$

(If DE is zero, we have $Fg = \frac{1}{3}FC$ for the centre of gravity of the triangle ABC.)

9. Find the centre of gravity of the quadrant of an ellipse, whose equation is $a^2y^2 + b^2x^2 = a^2b^2$.

$$Ans.\ \bar{x} = \tfrac{4}{3}\frac{a}{\pi};\ \bar{y} = \tfrac{4}{3}\frac{b}{\pi}.$$

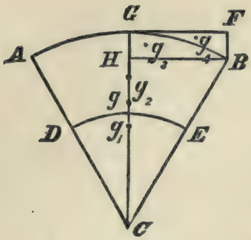


FIG. 48.

10. Find the centre of gravity of the circular sector ABC.

Let the angle $ACB = 2\theta$; then

$$\bar{x} = Cg = \frac{2}{3}\,\frac{r \sin\theta}{\theta}.$$

11. Find the centre of gravity of a part of a circular annulus $ABED$.

Let $AC = r$, $DC = r_1$, and $ACB = 2\theta$; then

$$Ans.\ \ Cg_2 = \frac{2}{3}\,\frac{\sin\theta}{\theta}\cdot\frac{r^2 + rr_1 + r_1^2}{r + r_1}.$$

12. Find the centre of gravity of the circular spandril FGB.

13. Find the centre of gravity of a circular segment.

$$Ans.\ \ Dist.\ from\ C = \frac{(chord)^3}{12\ area\ of\ segment}.$$

14. Find the distance of the centre of gravity of a complete cycloid from its vertex, r being the radius of the generating circle.

$$Ans.\ \ \bar{y} = \tfrac{7}{6} r.$$

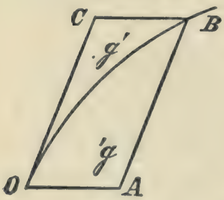


FIG. 49.

15. Find the centre of gravity of the parabolic spandril OCB, Fig. 49, in which $OC = y$, and $CB = x$.

$$Ans.\ \ \bar{x} = \tfrac{3}{10} x;$$
$$\bar{y} = \tfrac{3}{4} y.$$

16. Find the centre of gravity of a loop of the leminiscate, whose equation is $r^2 = a^2 \cos 2\theta$.

$$Ans.\ \ \bar{x} = \frac{\pi}{2^{\frac{5}{2}}}\,a.$$

17. Find the centre of gravity of a hemispherical surface.

$$Ans.\ \ \bar{x} = \tfrac{1}{2} r.$$

18. Find the centre of gravity of the surface generated by the revolution of a semi-cycloid about its base, a being the radius of the generating circle and $\bar{x}$ the distance from the vertex of the surface.

$$Ans.\ \ \bar{x} = \tfrac{26}{15} a.$$

19. The centre of gravity of the volume of a paraboloid of revolution is

$$\bar{x} = \tfrac{2}{3} x.$$

20. The centre of gravity of one half of an ellipsoid of revolution, of which the equation is $a^2y^2 + b^2x^2 = a^2b^2$, is

$$\bar{x} = \tfrac{3}{8} a.$$

21. The centre of gravity of a rectangular wedge is

$$\bar{x} = \tfrac{1}{3} a.$$

22. The centre of gravity of a semicircular cylindrical wedge, whose radius is r, is

$$\bar{x} = \tfrac{3}{16}\pi r.$$

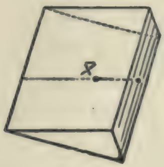


FIG. 50.

FIG. 51.

23. The vertex of a right circular cone is in the surface of a sphere, the axis of the cone passing through the centre of the sphere, the base of the cone being a portion of the surface of the sphere. If 2θ be the vertical angle of the cone, required the distance of the centre of gravity from the apex.

$$\textit{Ans.} \quad \frac{1 - \cos^6\theta}{1 - \cos^4\theta} r.$$

24. Find the distance from G, Fig. 48, to the centre of gravity of a spherical sector generated by the revolution of a circular sector GCA, about the axis GC.

$$\textit{Ans.} \quad \tfrac{1}{4}(GC + \tfrac{3}{2}GH).$$

25. A circular hole with a radius r is cut from a circular disc whose radius is R; required the centre of gravity of the remaining part, when the hole is tangent to the circumference of the disc.

26. Find the centre of gravity of the frustum of a pyramid or cone.

It will be in the line which joins the centre of gravity of the upper and lower bases. Let h be the length of this line, and a and b be corresponding lines in the lower and upper bases respectively, required the distance, measured on the line h, of the centre from the lower end.

$$\textit{Ans.} \quad \bar{x} = \tfrac{1}{4}h\,\frac{a^2 + 2ab + 3b^2}{a^2 + ab + b^2}.$$

If $b = 0$, we have the distance of the centre of a pyramid or cone from the base equal to $\frac{1}{4}h$.

27. Find the centre of gravity of the octant of a sphere in which the density varies directly as the nth power of the distance from the centre, r being the radius of the sphere.

$$\textit{Ans.} \quad \bar{x} = \frac{n+3}{2n+8}r = \bar{y} = \bar{z}.$$

28. Find the centre of gravity of a paraboloid of revolution of uniform density whose axis is a.

$$\textit{Ans.} \quad \bar{x} = \tfrac{2}{3}a.$$

SOME GENERAL PROPERTIES OF THE CENTRE OF GRAVITY.

77. *When a body is at rest on a surface, a vertical through the centre of gravity will fall within the support.*

For, if it passes without the support, the reaction of the surface upward and of the weight downward form a statical couple, and rotation will result.

78. *When a body is suspended at a point, and is at rest, the centre of gravity will be vertically under the point of suspension.*

The proof is similar to the preceding. When the preceding conditions are fulfilled the body is in equilibrium.

79. A body is in a condition of *stable equilibrium* when, if its position be slightly disturbed, it tends to return to its former position when the disturbing force is removed; of *unstable equilibrium* if it tends to depart further from its position of rest when the disturbing force is removed; and of *indifferent equilibrium* if it remains at rest when the disturbing force is removed.

EXAMPLE.

A paraboloid of revolution rests on a horizontal plane; required the inclination of its axis.

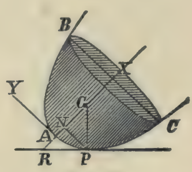


FIG. 52.

Let P be the point of contact of the paraboloid and plane, then will the vertical through P pass through the centre of gravity G, and PG will be a normal to the paraboloid.

The equation of a vertical section through the centre is $y^2 = 2px$, in which x is the axis, the origin being at the vertex.

Let $a = AX =$ the altitude of the paraboloid;

$\theta = GRP =$ the inclination of the axis;

then, $AG = \frac{2}{3}a$, (see example 28 on the preceding page);

$AN = \frac{2}{3}a - p$;

hence,

$$\tan\theta = \frac{dy}{dx} = \frac{p}{y} = \sqrt{\frac{p}{2x}} = \sqrt{\frac{p}{2(\frac{2}{3}a - p)}}$$

which will be positive and real as long as $\frac{2}{3}a$ exceeds p. In

this case the equilibrium is *stable.* When $\frac{4}{3}a$ exceeds p it will also rest on the apex, but the equilibrium for this position is *unstable.* When $\frac{4}{3}a = p$, $\theta = 90°$, and the segment will rest only on the apex. When $\frac{4}{3}a$ is less than p, $\tan\theta$ becomes imaginary, and hence, this analysis fails to give the position of rest; but by independent reasoning we find, as before, that it will rest on the apex, and that the equilibrium will be *stable.*

80. *In a plane material section the sum of the products found by multiplying each elementary mass by the square of its distance from an axis, equals the sum of the similar products in reference to a parallel axis passing through the centre, plus the mass multiplied by the square of the distance between the axes.*

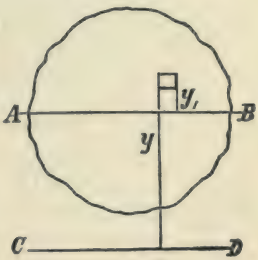


FIG. 53.

Let AB be an axis through the centre,

CD a parallel axis,

$D =$ the distance between AB and CD,

$dm =$ an elementary mass,

$y_1 =$ the ordinate from AB to m,

$y =$ the ordinate from CD to m, and

$M =$ the mass of the section.

Then

$$y^2 = (y_1 + D)^2 = y_1^2 + 2y_1 D + D^2.$$

Multiply by dm and integrate, and we have

$$\int y^2 dm = \int y_1^2 dm + 2D\int y_1 dm + D^2 \int dm.$$

But since AB passes through the centre, the integral of $y_1 dm$, when the whole section is included, is zero (see Eq. 71*b*), and $\int dm = M$; hence,

$$\int y^2 dm = \int y_1^2 dm + MD^2. \qquad (83)$$

Similarly, if dA be an elementary area, and A the total area, we have

$$\int y^2 dA = \int y_1^2 dA + AD^2.$$

81. *In any plane area, the sum of the products of each elementary area multiplied by the square of its distance from an axis, is least when the axis passes through the centre.*

This follows directly from the preceding equation, in which the first member is a minimum for $D = 0$.

CENTRE OF THE MASS.

82. *The centre of the mass is such a point that, if the whole mass be multiplied by its distance from an axis, it will equal the sum of the products found by multiplying each elementary mass by its distance from the same axis.*

Let $m =$ an elementary mass;

$M =$ the total mass;

x_1, y_1, and z_1 be the respective coördinates, of the centre of the mass, and

x, y, and z the general coördinates,

then, according to the definition, we have

$$\left.\begin{aligned} Mx_1 &= \Sigma mx; \\ My_1 &= \Sigma my; \\ Mz_1 &= \Sigma mz; \end{aligned}\right\} \qquad (84)$$

which being the same as (70) shows that when we consider the force of gravity as constant for all the particles of a body, the centre of the mass coincides with the centre of gravity. This is *practically* true for finite bodies on the surface of the earth, although the centre of gravity is *actually* nearer the earth than the centre of the mass is.

If the origin of coördinates be at the centre of the mass, we have

$$\Sigma mx = 0; \quad \Sigma my = 0; \quad \Sigma mz = 0; \qquad (84a)$$

which are the same as (71a).

CHAPTER IV.

NON-CONCURRENT FORCES.

83. Equilibrium of a solid body acted upon by any number of forces applied at different points and acting in different directions.

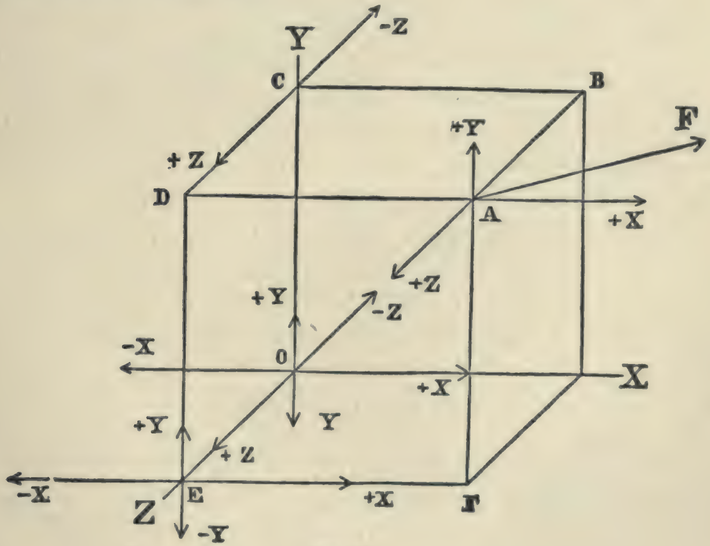


Fig. 54.

Let A be any point of a body, at which a force F is applied, and O the origin of coördinates, which, being chosen arbitrarily, may be within or without the body. On the coördinate axes construct a parallelopipedon having one of its angles at O, and the diagonally opposite one at A.

Let the *typical force* F be in the first angle and acting away from the origin, so that all of its direction-cosines will be positive; then will the sign of the axial component of any force be the same as that of the trigonometrical cosine of the angle which the direction of the force makes with the axis.

Let $\alpha =$ the angle between F and the axis of x,
$\beta =$ " " " " " " " " y,
$\gamma =$ " " " " " " " " z,
then will the X, Y, and *Z-components* of the force F be

$$X = F \cos \alpha,$$
$$Y = F \cos \beta,$$
$$Z = F \cos \gamma.$$

The point of application of the *X-component*, being at any point in its line of action, may be considered as at D, where its action-line meets the plane yz. At E introduce two equal and opposite forces, each equal and parallel to X, and since they will equilibrate each other, the mechanical effect of the system will be the same as before they were introduced. Combining the force $+ X$ at D with $- X$ at E, we have a *couple* whose arm is $DE = y =$ the *y-ordinate* of the point A. This couple, according to Article 54, will be negative, hence, its moment is

$$- Xy.$$

Hence, a force $+ X$ at A produces the same effect upon a body as the couple $- Xy$, and a force $+ X$ at E.

At the origin O introduce two equal and opposite forces, each equal to X, acting along the axis of x. This will not change the mechanical effect of the system. Combining $- X$ at O with $+ X$ at E, we have the couple $+ Xz$, and the force $+ X$ remaining at O. Hence, *a single force* $+X$ *at* A *is equivalent to an equal parallel force at the origin of coördinates, and the two couples,*

$$- Xy \text{ and } + Xz.$$

Treating the *Y-component* in a similar manner, we have the force

$+ Y$ *at the origin*, and

the moments,

$$+ Yx \text{ and } - Yz;$$

and similarly for the *Z-component*, the force

$+ Z$ *at the origin*, and

the moments,

$$- Zx \text{ and } + Zy.$$

But the couples $+ Zy$ and $- Yz$, have the common axis x,

and hence are equivalent to a single couple which is equal to the algebraic sum of the two; and similarly for the others; hence, the *six* couples may be reduced to the *three* following:

$$\begin{array}{lll} Zy - Yz, & \text{having } x \text{ for an axis}; \\ Xz - Zx, & \text{" } \quad y \text{ " " "}; \\ Yx - Xy, & \text{" } \quad z \text{ " " "}; \end{array}$$

hence, *for the single force F acting at A there may be substituted the three axial components of the force acting at the origin of coördinates, and three pairs of couples having for their axes the respective coördinate axes.*

If there be a *system* of forces, in which

F_1, F_2, F_3, etc., are the forces,

x_1, y_1, z_1, the coördinates of the point of application of F_1,

x_2, y_2, z_2, " " " " " " " " F_2,

etc., etc., etc.,

a_1, a_2, a_3, etc., the angles made by F_1, F_2, etc., respectively with the axis of x,

β_1, β_2, β_3, etc., the angles made by the forces with y, and

γ_1, γ_2, γ_3, etc., the corresponding angles with z;

then resolving each of the forces in the same manner as above, we have the *axial components*

$$\left.\begin{array}{l} X = F_1 \cos a_1 + F_2 \cos a_2 + F_3 \cos a_3 + \text{etc.} = \Sigma F \cos a; \\ Y = F_1 \cos \beta_1 + F_2 \cos \beta_2 + F_3 \cos \beta_3 + \text{etc.} = \Sigma F \cos \beta; \\ Z = F_1 \cos \gamma_1 + F_2 \cos \gamma_2 + F_3 \cos \gamma_3 + \text{etc.} = \Sigma F \cos \gamma; \end{array}\right\} \quad (85)$$

and the *component moments*

$$\left.\begin{array}{l} Zy - Yz = \Sigma(Fy \cos \gamma - Fz \cos \beta) = L; \\ Xz - Zx = \Sigma(Fz \cos a - Fx \cos \gamma) = M; \\ Yx - Xy = \Sigma(Fx \cos \beta - Fy \cos a) = N; \end{array}\right\} \qquad (86)$$

in which L, M, and N are used for brevity.

RESULTANT FORCE AND RESULTANT COUPLE.

84. Let R = the resultant of a system of forces concurring at the origin of coördinates, and having the same magnitudes and directions as those of the given forces;

a, b, and c = the angles which it makes with the axes x, y, and z respectively;

G = the moment of the resultant couple;

d, e, and f = the angles which the axis of the resultant couple makes with the axes x, y, and z respectively;

then

$$\left.\begin{aligned} X &= R\cos a; \\ Y &= R\cos b; \\ Z &= R\cos c; \end{aligned}\right\} \qquad (87)$$

$$\left.\begin{aligned} L &= G\cos d; \\ M &= G\cos e; \\ N &= G\cos f. \end{aligned}\right\} \qquad (88)$$

If a force and a couple, equal and opposite respectively to the resultant force and resultant couple, be introduced into the system, there will be equilibrium, and R and G will both be zero. Hence, for equilibrium, we have

$$X = 0; \qquad Y = 0; \qquad Z = 0; \qquad (89)$$

$$L = 0; \qquad M = 0; \qquad N = 0. \qquad (90)$$

85. Discussion of equations (87) and (88).

1. *Suppose that the body is perfectly free to move in any manner.*

a. If the forces concur and are in equilibrium, equations (87) only are necessary, and are the same as equations (60); hence, we will have

$$X = 0, \quad Y = 0, \quad Z = 0.$$

b. If $R = 0$ and G is finite, equations (88) only are necessary.

c. If R and G are both finite, then all of equations (87) and (88) may be necessary.

2. If *one point* of the body is *fixed*, there can be no translation, and equations (88) will be sufficient.

3. If an *axis parallel to x is fixed* in the body, there may be translation along that axis, and rotation about it: hence, the 1st of (87) and the 1st of (88) are sufficient.

4 If *two points are fixed*, it cannot translate, but may rotate; and by taking x so as to pass through the two points, the equation $L = 0$ is sufficient.

5. If *one point only* is confined to the *plane* xy, the body will have every degree of freedom except moving parallel to z, and hence, all of equations (87) and (88) are necessary except the 3d of (87).

6. If *three points*, not in the same straight line, are confined to the *plane* xy, it may rotate about z, but cannot move parallel to z; hence, the 1st and 2d of (87) and the 3d of (88) are necessary and sufficient.

7. If *two axes parallel to* x are fixed, the body can move only parallel to x, and the 1st of (87) is sufficient.

8. If the *forces* are parallel to the axis of y, there can be translation parallel to y only, and rotation about x and z.

9. If the forces are in the plane xy, the equations for equilibrium become

$$\left.\begin{aligned} X = \Sigma F \cos \alpha = R \cos a &= 0; \\ Y = \Sigma F \cos \beta = R \cos b &= 0; \\ Yx - Xy = \Sigma (Fx \cos \beta - Fy \cos \alpha) &= 0. \end{aligned}\right\} \qquad (91)$$

[OBS. In a mechanical sense, whatever holds a body is a force. Hence, when we say "a point is fixed," or, "an axis is fixed," it is equivalent to introducing an indefinitely large resisting force. Instead of finding the value of the resistance, it has, in the preceding discussion, been eliminated. When we say "the body cannot translate," it is equivalent to saying that finite active forces cannot overcome an infinite resistance.]

86. Applications of equations (91).

a. PROBLEMS IN WHICH THE TENSION OF A STRING IS INVOLVED.

1. *A body AB, whose weight is W, rests at its lower end upon a perfectly smooth horizontal plane, and at its upper end against a perfectly smooth vertical plane: the lower end is prevented from sliding by a string CB. Determine the tension on the string, and the pressure upon the horizontal and vertical planes.*

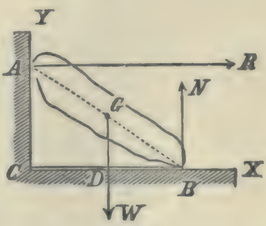


FIG. 55.

Take the origin of coördinates at C, the axis of x coinciding with CB, and y with AC, x being positive to the right, and y positive upwards.

Let $W =$ the weight of the body whose centre of gravity is at G;

$R =$ the reaction of the vertical wall, and, since there is no friction, its direction will be perpendicular to AC;

$N =$ the reaction of the horizontal plane, which will be perpendicular to CB;

$l =$ the horizontal distance from C to the vertical through the centre of gravity;

$t =$ the tension of the string.

The forces may be considered positive, and the sign of the component of the force will be that of the trigonometrical function. To determine the angle between the axis of $+x$ and the force, conceive a line drawn *from* the origin of coordinates parallel to and in the direction of the force, then will the angle be that swept over by a line from $+x$ turning left-handed to the line thus drawn. The origin of coördinates may be at any point, and the origin of moments at any other point.

Taking the origin of coördinates, and the origin of moments both at C, we have

$$X = R\cos 0^\circ + t\cos 180^\circ + W\cos 270^\circ + N\cos 90^\circ = 0;$$
$$Y = R\sin 0^\circ + t\sin 180^\circ + W\sin 270^\circ + N\sin 90^\circ = 0;$$
$$Moments = -R.AC + t.0 - W.CD + N.CB = 0.$$

Reducing gives

$$R - t = 0,$$
$$-W + N = 0,$$
$$-R.AC - W.CD + N.CB = 0;$$

or,

$$R = t,$$
$$W = N,$$
$$R = \frac{CB - CD}{AC}W = \frac{DB}{CA}W = t.$$

We see from this that the horizontal plane supports the entire weight of the piece, and that the pressure against the wall equals the tension of the string.

We also notice that the forces N and W being equal, parallel and opposite, constitute a couple whose arm is DB; and this must be in equilibrium with the couple t, CA, R; the arm being CA, hence we have $W.DB = t.CA$, as before.

2. A ladder rests on a smooth horizontal plane and against a vertical wall, the lower end being held by a horizontal string; a person ascends the ladder, required the pressure against the wall for any position on the ladder.

3. A uniform beam, whose length is AB and weight W, is held in a horizontal position by the inclined string CD, and carries a weight P at the extremity; required the tension of the string.

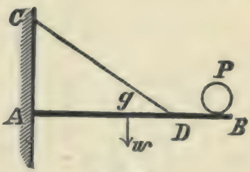


FIG. 56.

$$Ans.\ t = \frac{AB}{AD} \cdot \frac{DC}{AC}\,(P + \tfrac{1}{2}W).$$

4. A prismatic piece AB is permitted to turn freely about the lower end A, and is held by a string CE; given the position of the centre of gravity, the weight W of the piece, the inclination of the piece and string, and the point of attachment E; required the tension of the string, and the pressure against the lower end of the beam at A.

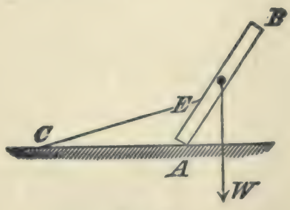


FIG. 57.

5. *A heavy piece AB is supported by two cords which pass over pulleys C and D, and have weights P_1 and P attached to them; required the inclination to the horizontal of the line AB joining the points of attachment of the cord.*

(Consider the pulleys as reduced to the points C and D.)

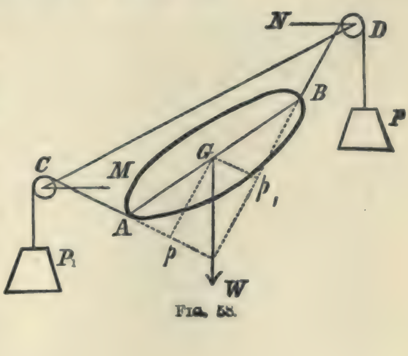


FIG. 58.

Let G, the centre of gravity of AB, be on the line joining the points of attachment A and B;

$a = AG$; $b = BG$;

i = the angle DCM;

δ = the inclination of BA to DC;

$\alpha = DCA$; and $\beta = CDB$.

Resolving horizontally and vertically, we have

$$X = P_1 \cos (180° - MCA) + P \cos NDB + W \cos 270° = 0;$$

$$= -P_1 \cos (\alpha - i) + P \cos (\beta + i) = 0; \qquad (a)$$

$$Y = P_1 \sin (\alpha - i) + P \sin (\beta + i) - W = 0. \qquad (b)$$

Taking the origin of moments at G, we have

$$-P_1 \times Gp + P \times Gp_1 + W \times 0 = 0;$$

or, $$-P_1 \,.\, a \sin (\alpha + \delta) + P \,.\, b \sin (\beta - \delta) = 0. \qquad (c)$$

The angle i is given by the conditions of the problem; hence the three equations (a), (b), and (c) are sufficient to determine the angles α, β, and δ, when the numerical values of the given quantities are known. The inclination will be $\delta + i$.

6. Suppose, in Fig. 58, that the strings are fastened at C and D, and that DC, AC, and BD are given, required the inclination of AB.

[The solution of this problem involves an equation of the 8th degree].

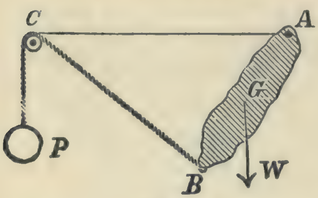


FIG. 59.

7. A heavy piece AB, Fig. 59, is free to swing about one end A, and is supported by a string BC which passes over a pulley at C, and is attached to a weight P; find the angle ACB when they are in equilibrium.

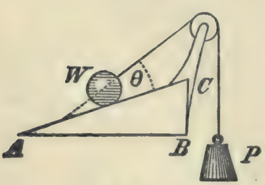


FIG. 60

8. A weight W rests on a plane whose inclination to the horizontal is i, and is held by a string whose inclination to the plane is θ; required the relation between the tension P and the weight, and the value of the normal pressure upon the plane.

Ans. $P = \dfrac{\sin i}{\cos \theta} W$; *Normal pressure* $= \dfrac{\cos (\theta + i)}{\cos \theta} W$.

b. EQUILIBRIUM OF PERFECTLY SMOOTH BODIES IN CONTACT WITH EACH OTHER.

9. *A heavy beam rests on two smooth inclined planes, as in Fig.* 61; *required the inclination of the beam to the horizontal, and the reactions of the respective planes.*

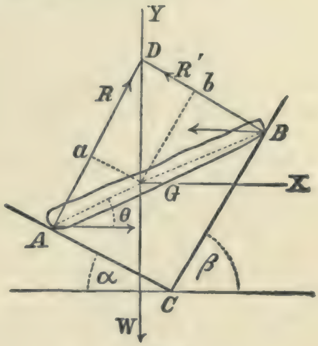


FIG. 61.

Let AC and CB be the inclined planes; AB the beam whose centre of gravity is at G. When it *rests*, the reactions of the planes must be normal to the planes, for otherwise they would have a component parallel to the planes which would produce motion.

Let $a_1 = AG$; $a_2 = GB$;
$R =$ the reaction at A;
$R' =$ " " " B;
$W =$ the weight of the beam;
$\alpha =$ the inclination of AC to the horizon;
$\beta =$ " " " BC " " " ;
$\theta =$ " " " AB " " " .

Take the origin of coördinates at the centre of gravity G of the body, x horizontal and y vertical.

The forces resolved horizontally give

$$X = R\cos(90° - \alpha) + R'\cos(90° + \beta) + W\cos 270° = 0; \quad (a)$$

and vertically,

$$Y = R\sin(90° - \alpha) + R'\sin(90° + \beta) + W\sin 270° = 0. \quad (b)$$

The moment of $R\sin\alpha$ is, . . . $+ R\sin\alpha \times AG\sin\theta$.
" " " $R'\sin\beta$ is, . . . $+ R'\sin\beta \times GB\sin\theta$.
" " " $W\cos 90°$ is, . . . 0.
" " " $R\cos\alpha$ is, . . . $- R\cos\alpha \times AG\cos\theta$.
" " " $R'\cos\beta$ is, . . . $+ R'\cos\beta \times GB\cos\theta$

$$\begin{aligned} \text{Hence } Xy - Yx &= Ra_1\sin\alpha\sin\theta + R'a_2\sin\beta\sin\theta \\ &\quad - Ra_1\cos\alpha\cos\theta + R'a_2\cos\beta\cos\theta \\ &= -Ra_1\cos(\alpha+\theta) + R'a_2\cos(\beta-\theta) = 0. \quad (c) \end{aligned}$$

It is generally better to deduce the values of the moments directly from the definitions; (see Articles 51 to 57). To do this in the present case, let fall from G the perpendiculars aG and bG upon the action-lines of the respective forces; then

$$bG = a_2 \sin(90° - (\beta - \theta)) = a_2 \cos(\beta - \theta);$$
$$aG = a_1 \sin(90° - (\alpha + \theta)) = a_1 \cos(\alpha + \theta);$$

and we have

the moments $= -R \cdot aG + R' \cdot bG = -Ra_1 \cos(\alpha + \theta) + R'a_2 \cos(\beta - \theta) = 0$; as given above.

Solving equations (a), (b), and (c), we find

$$R = \frac{\sin\beta}{\sin(\alpha+\beta)} W; \qquad R' = \frac{\sin\alpha}{\sin(\alpha+\beta)} W;$$

$$\tan\theta = \frac{a_1 \cos\alpha \sin\beta - a_2 \sin\alpha \cos\beta}{(a_1 + a_2)\sin\alpha \sin\beta}.$$

If $R = R'$, then

$$\sin\beta = \sin\alpha;$$

which are the conditions necessary to make the normal reactions equal to each other.

The reactions prolonged will meet the vertical through the centre of gravity at a common point D, and if the beam be suspended at D by means of the two cords DA and DB it will retain its position when the planes AC and CB are removed.

If $\beta = 90°$, the plane CB will be vertical, and we find

$$R = W \sec\alpha; \qquad R' = \frac{\sin\alpha}{\cos\alpha} W = W \tan\alpha;$$

$$\tan\theta = \frac{a_1}{a_1 + a_2} \cot\alpha.$$

If $a_1 = a_2$, then

$$\tan\theta = \frac{\sin(\beta - \alpha)}{2 \sin\alpha \sin\beta}.$$

If $\beta = 90°$ and $\alpha = 0°$, then

$$R' = 0, \ \theta = 90°, \text{ and } R = W.$$

A special case is that in which the beam coincides with one of the planes. The formulas do not apply to this case.

10. Two equal, smooth cylinders rest on two smooth planes whose inclinations are α and β respectively; required the inclination, θ, of the line joining their centres.

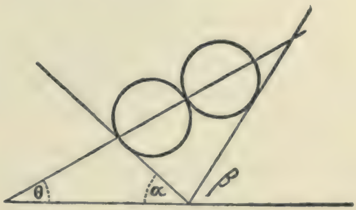


FIG. 62.

Ans. $\tan\theta = \frac{1}{2}(\cot\alpha - \cot\beta)$.

11. A heavy, uniform, smooth beam *rests on* one edge of a box at C, and *against* the vertical side opposite; required its inclination to the vertical. Let g be the centre of gravity.

Ans. $\sin\theta = \sqrt[3]{\dfrac{DE}{Bg}}$.

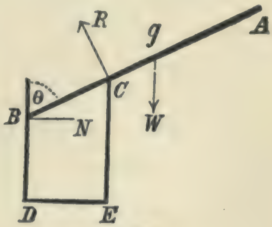


FIG. 63.

12. Three equal, smooth cylinders are placed in a box, the two lower ones being tangent to the sides of the box and to each other, and the other placed above them and tangent to both; required the pressure against the bottom and sides of the box.

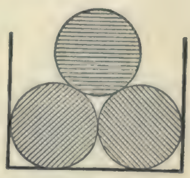


FIG. 64.

Ans. Pressure on the bottom = total weight of the cylinders.

Pressure on one side $= \frac{1}{2}$ *weight of one cylinder* $\times$ tan 30°.

13. Two homogeneous, smooth, prismatic bars rest on a horizontal plane, and are prevented from sliding upon it; required their position of equilibrium when leaning against each other.

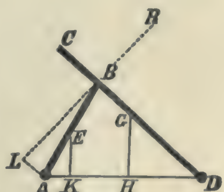


FIG. 65.

Let AB and CD be the two bars, resting against each other

at B; then will they be in equilibrium when the resultant of their pressures at B is perpendicular to the face of CD.

Let $b = AB$; $c = CD$; $x = BD$;
$a = AD =$ the distance between the lower ends of the bars;
$W =$ the weight of AB;
$W_1 =$ the weight of CD;
E and G the respective centres of gravity of the bars, which will be at the middle of the pieces; then we have

$$2(a^2 + b^2 - x^2)x^3 W = c(a^2 - b^2 + x^2)(-a^2 + b^2 + x^2) W_1;$$

which is an equation of the fifth degree, and hence always admits of one real root.

14. *The upper end of a heavy piece rests against a smooth, vertical plane, and the lower end in a smooth, spherical bowl; required the position of equilibrium.*

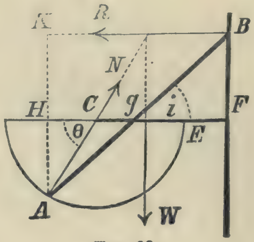


Fig. 66.

Let AB be the piece, BF the vertical surface, EA the spherical surface, and g the centre of gravity of the piece. When it is in equilibrium, the reaction at the lower end will be in the direction of a normal to the surface, and hence will pass through C, the centre of the sphere, and the reaction of the vertical plane will be horizontal.

Let $W =$ the weight of the piece;
$r =$ the radius of the sphere;
$a = Ag$; $b = Bg$; $l = AB$; $d = CF$;
$R =$ the reaction of the vertical plane;
$N =$ the reaction of the spherical surface;
$i =$ the inclination of the beam to the horizontal;
$\theta =$ the inclination of the radius to the horizontal.

Take the origin of coördinates at g, x horizontal and y vertical; and we have

$$X = N\cos\theta + R\cos 180° + W\cos 270° = 0;$$
$$Y = N\sin\theta + R\sin 180° + W\sin 270° = 0;$$

$$\textit{Moments} = +R.b\sin i - N.a\sin(\theta - i) = 0;$$

and the geometrical relations give,

$$l \cos i = KB = HF = d + r \cos \theta.$$

From these equations, we have

$$N = W \operatorname{cosec} \theta; \; R = W \cot \theta;$$
$$a \sin (\theta - i) - b \cos \theta \sin i = 0,$$

which, by developing and reducing, becomes

$$(a + b) \tan i = a \tan \theta;$$

this, combined with the fourth equation above, will determine i and θ.

The position is independent of the weight of the piece, but depends upon the position of its centre of gravity.

15. A heavy prismatic bar of infinitesimal cross-section rests against the concave arc of a vertical parabola, and a pin placed at the focus; required the position of equilibrium.

Let $l = AB =$ length of the bar; $p = CD =$ one-fourth the parameter of the parabola, C being the focus, and $\theta = ACD$.

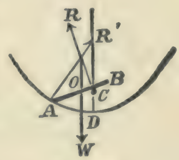


FIG. 67.

$$Ans. \; \theta = 2 \cos^{-1} \left(\frac{p}{l}\right)^{\frac{1}{3}}.$$

16. Required the form of the curve such that the bar will rest in all positions.

Ans. The polar equation is $r = \frac{1}{2}l + c \sec \theta$, in which l is the length of the bar, and c an arbitrary constant. It is the equation of the conchoid of Nicomedes.

c. INDETERMINATE PROBLEMS.

17. *To determine the pressures exerted by a door upon its hinges.*

Let $W =$ the weight of the door;
$a =$ the distance between the hinges;
$b =$ the horizontal distance from the centre of gravity of the door to the vertical line which passes through the hinges;
$F =$ the vertical reaction of the upper hinge;
$F_1 =$ the vertical reaction of the lower hinge;
$H =$ the horizontal reaction of the upper hinge;
$H_1 =$ the horizontal reaction of the lower hinge;

then

$$X = H - H_1 = 0;$$
$$Y = F + F_1 - W = 0;$$
$$Xy - Yx = Ha - Wb = 0;$$

which give

$$H = H_1 = \frac{b}{a} W; \text{ and}$$

$$F + F_1 = W.$$

The result, therefore, is indeterminate, but we can draw two general inferences: 1st, *The horizontal pressures upon the hinges are equal to each other but in opposite directions;* and, 2d, *The vertical reaction upon both hinges equals the weight of the door.*

It is necessary to have additional data in order to determine the *actual pressure* on each hinge. The ordinary imperfections of workmanship will cause one to sustain more weight than the other, but as they wear they may approach an equality.

The horizontal and vertical pressures being known, the actual pressures may be found by the *triangle of forces.* If the upper end sustains the whole weight, the total pressure upon it will be $\frac{W}{a}\sqrt{a^2 + b^2}$. If each sustains one-half the weight, the pressure on each will be one-half this amount.

18. A rectangular stool rests on four legs, one being at each corner of the stool; required the pressure on each.

(The data are insufficient.)

19. A weight P is supported by three unequally inclined struts in one plane; required the amount which each will sustain.

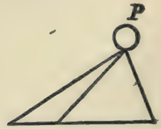


FIG. 68.

[OBS. If more conditions are given than there are quantities to be determined, they will either be redundant or conflicting.]

d. STRESS ON FRAMES.

20. *Suppose that a triangular truss, Fig.* 69, *is loaded with*

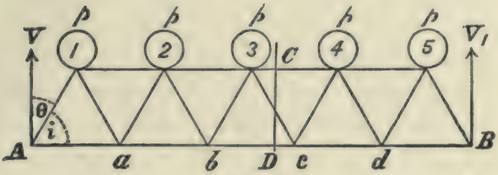


FIG. 69.

equal weights at the upper apices; it is required to find the stress upon any of the pieces of the truss.

[The *stress* is the pull or push on a piece.]

Let the truss be supported at its ends, and let

$l = Aa = ab =$ etc., $=$ the equal divisions of the span AB;
$N =$ the number of bays in the chord AB;
$L = Nl = AB$, the span;
p_1, p_2, p_3, etc., be the weights on the successive apices; which we will suppose are equal to each other; hence
$p = p_1 = p_2 =$ etc.;
$Np =$ the total load;
$V =$ the reaction at A; and
$V_1 =$ " " " B.

1st. *There will be equilibrium among the external forces.*

All the forces being vertical, their horizontal components will be zero, hence

$$X = 0;$$
$$Y = V + V_1 - \Sigma p = V + V_1 - Np = 0; \qquad (a)$$

and taking the origin of moments at B, observing that the moment of the load is the total load multiplied by the horizontal distance of its centre of gravity from B, we have

$$- V \cdot AB + Np \cdot \tfrac{1}{2}AB = 0;$$

or,

$$V \cdot L - Np \cdot \tfrac{1}{2}L = 0;$$
$$\therefore V = \tfrac{1}{2}Np;$$

which in (a) gives V_1 also equal to $\frac{1}{2}\Lambda p$; hence the supports sustain equal amounts, as they should, since the load is symmetrical in reference to them, and is independent of the form of trussing.

2d. *To determine the internal forces.—Conceive that the truss is cut by a vertical plane and either part removed while we consider the remaining part. To the pieces in the plane section, apply forces acting in such a manner as to produce the same strains as existed before they were severed. Consider the forces thus introduced as external, and the problem is reduced to that of determining their value so that there shall be equilibrium among the new system of external forces.*

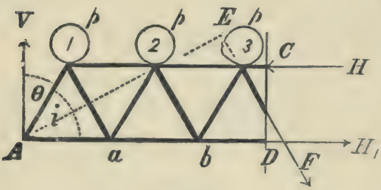


Fig. 70.

Let CD, Fig. 69, be a vertical section, and suppose that the right-hand part is removed. Introduce the external forces in place of the strains, as shown in Fig. 70.

Let H = the compressive strain in the upper chord;
H_1 = the tensile strain in the lower chord;
F = the pull in the inclined piece;
θ = the inclination of F to the vertical;
n = the number of the bay, bD, counting from A (which in the figure is the 3d bay); and
$D = CD$ = the depth of the frame.

The origin of coördinates may be taken at any point. Take it at A, x being horizontal and y vertical.

Resolving the forces,

$$X = H_1 \cos 0^\circ + H \cos 180^\circ + F \cos (270^\circ + \theta^\circ) + V \cos 90^\circ + \Sigma p \cos 270^\circ = 0;$$
$$Y = H_1 \sin 0^\circ + H \sin 180^\circ + F \sin (270^\circ + \theta^\circ) + V \sin 90^\circ + \Sigma p \sin 270^\circ = 0;$$

or,
$$H_1 - H + F \sin \theta = 0; \qquad (a)$$
$$V - np - F \cos \theta = 0; \qquad (b)$$

and the moments,
$$-\tfrac{1}{2}n^2pl + HD - F \,.\, nl \cos \theta = 0. \qquad (c)$$

Eliminating F between equations (b) and (c), substituting the value of $V = \frac{1}{2}Np$, and reducing, give

$$H = \frac{pl}{2D} n (N - n) ; \qquad (d)$$

that is, *the strains on the bays of the upper chord vary as the product of the segments into which the lower chord is divided by the joint directly under the bay considered.*

From (b) we have

$$F \cos \theta = V - np = \tfrac{1}{2}(N - 2n) p ; \qquad (e)$$

and since θ is constant, *the stress on the inclined pieces decreases uniformly from the end to the middle.*

At the middle $n = \frac{1}{2}N$, and $F = 0$; hence, *for a uniform load, there is no stress on the central braces.*

If F were considered as a *push*, equation (e) would be negative.

Eliminating H and F from (a), we have

$$H_1 = \left\{ N \ (n - \tfrac{1}{2}) - n (n - 1) \right\} \frac{pl}{2D} \qquad (f)$$

For forces in a plane the conditions of statical equilibrium give only three independent equations, (a), (b) and (c); (or Eqs. (91)); hence, *if a plane section cuts more than three independent pieces in a frame, the stresses in that section are indeterminate*, unless a relation can be established among the stresses, or a portion of them be determined by other considerations.

21. If $N = 7, p = p_1 = p_2 =$ etc. $= 1{,}000$ lbs., $AB = 56$ feet and $D = 4$ feet ; required the stress on each piece of the frame.

22. In Fig. 69, if p_1 and p_2 are removed, and $p_3 = p_4 = p_5 =$ 1,000 lbs., find the stress on the bay $2 - 3$, and the tie $2 - b$.

23. If all the joints of the lower chord are equally loaded, and no load is on the upper chord, required the stress on the n^{th} pair of braces, counting from A, Fig. 69.

Ans. $\frac{1}{2}(N - 2n + 1) p \sec \theta$.

24. *A roof truss ADB is loaded with equal weights at the equidistant joints* 1, 2, 3, *etc.; required the stress on any of its members.*

[OBS. A load composed of equal weights on all the joints will produce the same stress as that of a load uniformly distributed, except that the latter would produce cross strains upon the rafters, which it is not our purpose to discuss in this place.]

Let the tie AB be divided into equal parts, Aa, ab, etc., and the joints connected as shown in the figure. The joints are

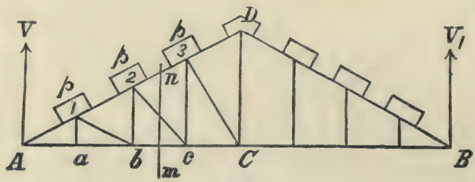


FIG. 71.

assumed to be perfectly flexible. The right half of Fig. 71 may be trussed in any manner by means of ties or braces, or both, and yet not affect the analysis applied to the left half.

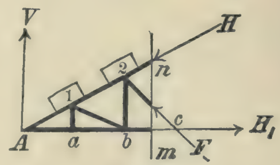


FIG. 72.

Conceive a vertical section nm and the right-hand part removed. Introduce the forces H, H_1 and F as previously explained, and the conditions of the problem will be represented by Fig. 72. The letters of reference given below involve both figures.

Let $N =$ the number of equal divisions (bays) in AB;
$n =$ the number of the bay bc counting from A;
$l = Aa = ab$, etc.;
$p =$ the weight on any one of the joints of the rafter;
$V =$ the vertical reaction at A or B;
$D = DC$, the depth at the vertex;
$\theta = b2c$; and $i = DAC$.

Then

$$(N-1)\,p = \text{the total load};\ \therefore\ V = \tfrac{1}{2}(N-1)\,p.$$

Take the origin of coördinates at A, and the *origin of moments* at the joint marked 2. Resolving the forces shown in Fig. 72 horizontally and vertically, we have

$$X = H\cos(180^\circ + aA1) + H_1\cos 0^\circ + F\sin(-b2c) = 0;$$
$$Y = V - (n-1)p + H\sin(180^\circ + aA1) + H_1\sin 0^\circ + F\cos(-b2c)$$
$$= 0;$$

or,
$$-H\cos i + H_1 - F\sin\theta = 0;$$
$$V - (n-1)p - H\sin i + F\cos\theta = 0;$$

also the *moments*,

$$H_1 b2 - V.Ab + (n-1)p.\tfrac{1}{2}(n-2)\,l = 0.$$

But from Fig. 71 we have

$$\frac{b2}{CD} = \frac{Ab}{AC} = \frac{(n-1)\,l}{\frac{1}{2}Nl}.$$

Substituting in the equation of moments the value of $b2$ found above, of $V = \frac{1}{2}(N-1)p$, of $Ab = (n-1)\,l$, and reducing, give

$$H_1 = \frac{Nl}{4D}(N - n + 1)\,p.$$

By means of the other two equations, and $(n-1)\tan i \tan\theta = 1$, we find

$$H = \tfrac{1}{2}(N-n)\,p\,\operatorname{cosec} i;$$
$$F = \tfrac{1}{2}(n-1)\,p\sec\theta.$$

e. STRESS IN A LOADED BEAM.

25. *Suppose that a beam is firmly fixed in a wall at one end, and that the projecting end is loaded with a weight P; required the forces in a vertical section mn*, Fig. 73.

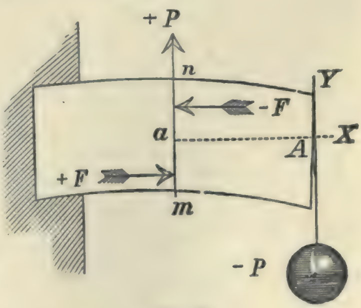


Fig. 73.

Take the origin of coördinates at A, x horizontal and y vertical. Take the plane section perpendicular to the axis of x. Without assuming to know the directions in which the

forces in the section act, we may conceive them to be resolved into horizontal and vertical components. Let F be the *typical* horizontal force, then will

$$X = \Sigma F = 0;$$

hence, some of the *F-forces* will be positive, and the others negative.

Neglecting the weight of the beam, and letting Y_1 be the sum of the vertical components in *nm*, we have

$$Y = Y_1 - P = 0 \quad \therefore Y_1 = P;$$

as shown in the figure.

The forces, $+P$ and $-P$, constitute a *couple* whose arm is Aa; and since the *F-forces* are the only remaining ones, the resultant of the $+F$'s and the $-F$'s must constitute a couple whose moment equals $P.Aa$ with a contrary sign.

[OBS. Investigations in regard to the distribution of the forces over the plane section belong to the *Resistance of Materials.*]

f. LOADED CORD.

26. *Suppose that a perfectly flexible, inextensible cord is fixed at two points and loaded continuously, according to any law; it is required to find the equation of the curve and the tension of the cord.*

Assuming that equilibrium has become established, we may treat the problem as if the cord were rigid, by considering the curve which it assumes as the locus of the point of application of the resultant. The

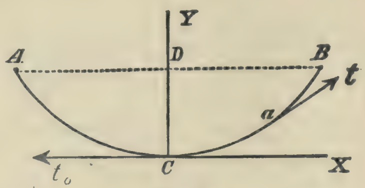


FIG. 74.

resultant at any point will be in the direction of a tangent to the curve at that point; for otherwise it would have a normal component which would tend to change the form of the curve.

Take the origin of coördinates at the lowest point of the curve. Let a be *any* point whose coördinates are x and y;

X = the sum of the *x-components* of all the external forces between the origin and a;

Y = the sum of the *y-components*;

$t =$ the tension of the cord at a;
$t_0 =$ the tension at the origin.

Resolving the tension (t) by multiplying it by the direction-cosine, we have

$$t\frac{dx}{ds} = \text{the } x\text{-component of } t, \text{ and}$$

$$t\frac{dy}{ds} = \text{the } y\text{-component.}$$

For the part Ca, equations (91) become

$$\left.\begin{aligned} -t_0 + X + t\frac{dx}{ds} &= 0; \\ Y + t\frac{dy}{ds} &= 0; \\ Xy - Yx &= 0. \end{aligned}\right\} \qquad (a)$$

[Obs. In the problems which we shall consider, the third of these equations will be unnecessary, since the other two furnish all the conditions necessary for solving them.]

Let all the applied forces be vertical.

Then $X = 0$, and the first two of equations (a) become

$$\left.\begin{aligned} -t_0 + t\frac{dx}{ds} &= 0; \\ Y + t\frac{dy}{ds} &= 0. \end{aligned}\right\} \qquad (b)$$

From the first of these we have

$$t\frac{dx}{ds} = t_0 = a \text{ constant};$$

hence, the *horizontal component of the tension will be constant throughout the length for any law of vertical loading.*

From the second of (b), we have

$$t\frac{dy}{ds} = -Y;$$

hence, the *vertical component of the tension at any point equals the total load between the lowest point and the point considered.*

27. *Let the load be uniformly distributed over the horizontal.*

(This is approximately the condition of the ordinary suspension bridge.)

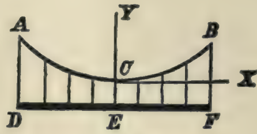


Fig. 75.

Let $w =$ the load per unit of length, then

$$Y = -wx;$$

and (b) becomes

$$\left.\begin{aligned} -t_0 + t\frac{dx}{ds} &= 0; \\ -wx + t\frac{dy}{ds} &= 0. \end{aligned}\right\} \qquad (c)$$

Eliminating t gives

$$t_0 dy = wx dx;$$

and integrating gives

$$t_0 y = \tfrac{1}{2}wx^2 + (C = 0);$$

$$\therefore\ x^2 = \frac{2t_0}{w}y; \qquad (d)$$

hence, the curve is a parabola whose axis is vertical, and whose parameter is $\frac{2t_0}{w}$. The parameter will be constant when $t_0 \div w$ is constant; hence *the tension at the lowest point will be the same for all parabolas having the same parameter and the same load per unit along the horizontal, and is independent of the length of the curve.*

To find the tension at the lowest point, substitute in equation (d) the value of the coördinates of some known point. Let the coördinates of the point A be x_1 and y_1; then (d) gives

$$t_0 = \frac{wx_1^2}{2y_1}. \qquad (e)$$

To find the tension at any point we have from the first of equations (c) and the Theory of Curves

$$t = t_0 \frac{ds}{dx} = t_0 \frac{\sqrt{dx^2 + dy^2}}{dx} = t_0 \sqrt{1 + \frac{dy^2}{dx^2}}$$

$$= \frac{wx_1^2}{2y_1}\sqrt{1 + \frac{dy^2}{dx^2}}. \qquad (f)$$

To find the tension at the highest point A, from (d) find

$$\frac{dy}{dx} = \frac{2y_1}{x_1}; \qquad (g)$$

substitute in (f), and we obtain

$$t_1 = \frac{wx_1}{2y_1}\sqrt{x_1^2 + 4y_1^2}.$$

(To find t_0 by the Theory of Moments, take the origin at A. The load on x_1 will be wx_1, and its arm the horizontal distance to the centre of gravity of the load, or $\frac{1}{2}x_1$; hence, its moment will be $\frac{1}{2}wx_1^2$. The moment of the tension will be t_0y_1; hence,

$$t_0y_1 = \tfrac{1}{2}wx^2_1 \text{ or } t_0 = \frac{wx_1^2}{2y_1}, \text{ as before.)}$$

The slope (or inclination of the curve to the horizontal) may be found from equation (g); which gives

$$\tan i = \frac{2y_1}{x_1}.$$

28. THE CATENARY. A catenary is the curve assumed by a perfectly flexible string of uniform section and density, when suspended at two points not in the same vertical. Mechanically speaking the load is uniformly distributed over the arc, and hence varies directly as the arc.

To find the equation,

let w = the weight of the cord per unit of length;

$\therefore Y = -ws$ (s being the length of the arc);

and equations (b) become

$$\left.\begin{aligned} -t_0 + t\frac{dx}{ds} &= 0; \\ -ws + t\frac{dy}{ds} &= 0. \end{aligned}\right\} \qquad (h)$$

Transposing and dividing the second by the first, gives

$$\frac{dy}{dx} = \frac{w}{t_0} s \cdot$$

and differentiating, substituting the value of ds and reducing, give

$$d\left(\frac{dy}{dx}\right) = \frac{w}{t_0} ds = \frac{w}{t_0} dx\sqrt{1 + \frac{dy^2}{dx^2}};$$

$$\therefore \frac{w}{t_0} dx = \frac{d\left(\frac{dy}{dx}\right)}{\sqrt{1 + \frac{dy^2}{dx^2}}}.$$

Integrating gives

$$\frac{w}{t_0} x = \log_e \left[\frac{dy}{dx} + \sqrt{1 + \frac{dy^2}{dx^2}}\right];$$

or, passing to exponentials, gives

$$e^{\frac{w}{t_0} x} = \frac{dy}{dx} + \sqrt{1 + \frac{dy^2}{dx^2}} = \frac{dy}{dx} + \frac{ds}{dx}; \qquad (i)$$

or,

$$1 + \frac{dy^2}{dx^2} = \left[e^{\frac{w}{t_0} x} - \frac{dy}{dx}\right]^2;$$

from which we find

$$\frac{dy}{dx} = \tfrac{1}{2}\left[e^{\frac{w}{t_0} x} - e^{-\frac{w}{t_0} x}\right]; \qquad (j)$$

which integrated gives

$$y = \tfrac{1}{2}\frac{t_0}{w}\left[e^{\frac{w}{t_0} x} + e^{-\frac{w}{t_0} x}\right] + \left(C = -\frac{t_0}{w}\right) \qquad (k)$$

$$= \tfrac{1}{2}\frac{t_0}{w}\left[e^{\frac{w}{2t_0} x} - e^{-\frac{w}{2t_0} x}\right]^2; \qquad (l)$$

which is the equation of the Catenary.

Eliminating $\frac{dy}{dx}$ between equations (i) and (j), we find

$$\frac{ds}{dx} = \tfrac{1}{2}\left[e^{\frac{w}{t_0} x} + e^{-\frac{w}{t_0} x}\right];$$

the integral of which is

$$s = \frac{1}{2}\,\frac{t_0}{w}\left[e^{\frac{w}{t_0}x} - e^{-\frac{w}{t_0}x}\right] + (C = 0); \qquad (m)$$

which gives the length of the curve.

The following equations may also be found

$$x = \frac{t_0}{w}\log_e\left\{\frac{ws}{t_0} + \sqrt{1 + \frac{w^2s^2}{t_0^2}}\right\};$$

$$t = \sqrt{t_0^2 + w^2s^2};$$

$$s = \frac{t_0}{w}\cdot\frac{dy}{dx}$$

If θ = the inclination of the curve to the vertical, then

$$x = s\tan\theta\log_e\cot\tfrac{1}{2}\theta.$$

The tensions, t and t_0, are so involved that they can be determined only by a series of approximations. The full development of these equations for practical purposes belongs to Applied Mechanics.

The catenary possesses many interesting geometrical and mechanical properties, among which we mention the following:—

The centre of gravity of the catenary is lower than for any other curve of the same length joining two fixed points.

If a common parabola be rolled along a straight line, the locus of the focus will be a catenary.

According to Eq. (k) it appears that if the origin of coördinates be taken directly below the vertex at a distance equal to $t_0 \div w$, the constant of integration will be zero. (This distance equals such a length of the cord forming the catenary as that its weight will equal the tension at the lowest point of the curve). A horizontal line through this point is the *directrix of the catenary.*

The radius of curvature at any point of the catenary equals the normal at that point, limited by the directrix.

The tension at any point equals the weight of the cord forming the catenary whose length equals the ordinate of the point from the directrix.

If an indefinite number of strings (without weight) be suspended from a catenary and terminated by a horizontal line, and the catenary be then drawn out to a straight line, the lower ends of the vertical lines will be in the arc of a parabola.

If the weight of the cord varies continuously according to any known law the curve is called *Catenarian.*

29. *To determine the equation of the Catenarian curve of uniform density in which the section varies directly as the tension.*

Let $k =$ the variable section;
$\delta =$ the weight of a unit of volume of the cord;
$c =$ the ratio of the section to the tension;

then

$$Y = -\int \delta k ds; \quad k = ct; \quad \therefore Y = -\delta c \int t ds;$$

which substituted in (b) and reduced, gives

$$\delta c y = \log_e \sec c\delta x,$$

for the required equation.

g. LAW OF LOADING.

30. *It is required to find the* LAW OF LOADING *so that the action-line of the resultant of the forces at any point shall be tangent to a given curve.*

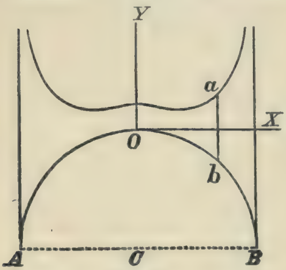


FIG. 76.

Assume the loading to be of uniform density, and the variations in the loading to be due to a variable depth. In Fig. 76, let O be the origin of coördinates; $Z = ab =$ the depth of loading over a point whose abscissa is x; $d =$ the depth of the loading over the rigin, and $\delta =$ the weight per unit of volume of the loading, then

$$Y = -\int \delta Z dx;$$

which in Eq. (b) gives

$$-t_0 + t\frac{dx}{ds} = 0;$$

$$-\delta \int Z\, dx + t\,\frac{dy}{ds} = 0;$$

Transposing, and dividing the latter by the former, gives

$$\frac{dy}{dx} = \frac{\delta}{t_0}\int Zdx;$$

which, differentiated, gives

$$\frac{d^2y}{dx^2} = \frac{\delta Z}{t_0}.$$

But, from the Theory of Curves, we have

$$\frac{d^2y}{dx^2} = \frac{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}{\rho} = \frac{\sec^3 i}{\rho},$$

in which ρ is the radius of curvature, and i is the angle between a tangent to the curve and the axis of x. From these we readily find

$$Z = \frac{t_0}{\delta}\frac{\sec^3 i}{\rho}.$$

At the origin $\rho = \rho_0$, $i = 0$, and $Z = d$; which values substituted in the preceding equation give

$$\frac{t_0}{\delta} = d\rho_0;$$

$$\therefore\ Z = d\rho_0 \frac{\sec^3 i}{\rho}. \qquad (n)$$

Discussion. For all curves which have a vertical tangent, we have at those points

$i = 90°$; $\therefore \sec i = \infty$, and, if ρ is finite

$$Z = \infty;$$

hence, it is practically impossible to load such a curve throughout its entire length in such a manner that the resultant shall be in the direction of the tangent to the curve. A portion of the curve, however, may be made to fulfil the required condition.

Let the given curve be the arc of a circle; then $\rho = \rho_0$, and equation (n) becomes

$$Z = d \sec^3 i,$$

from which the upper limit of the loading may be found. For

small angles $\sec^3 i$ will not greatly exceed unity and hence, the upper limit of the load will be nearly parallel to the arc of the circle for a short distance each side of the highest point. At the extremities of the semicircle, $i = 90°$, and $Z = \infty$.

If the given curve be a parabola, we find $Z = d$, that is, the depth of loading will be constant; or, in other words, uniformly distributed over the horizontal. This is the reverse of Prob. 27.

(The principles of this topic may be used in the construction and loading of arches.)

31. *Let the tension of the cord be uniform.*

We observe in this case that the loading must act normally to the curve at every point, for if it were inclined to it, the tangential component would increase or decrease the tension.

Let p = the *normal* pressure per unit of length of the arc; then pds = the pressure on an *element* of length, and this multiplied by the direction-cosine which it makes with the axis of x, and the expression integrated, give

$$\int pds\left(\frac{dx}{ds}\right) = \int pdx = \text{the } x\text{-component, and}$$

$$\int pdy = \text{the } y\text{-component of the pressures.}$$

hence, equations (a), p. 131, become

$$-t_0 + \int pdx + t\frac{dx}{ds} = 0;$$

$$\int pdy + t\frac{dy}{ds} = 0;$$

differentiating which, give

$$pdx + td\left(\frac{dx}{ds}\right) = 0;$$

$$pdy + td\left(\frac{dy}{ds}\right) = 0.$$

Transposing, squaring, adding and extracting the square root, give

$$p = t\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}^{\frac{1}{2}} = \frac{t}{\rho}; \qquad (o)$$

that is, *the normal pressure varies inversely as the radius of curvature.*

1. If a string be stretched upon a perfectly smooth curved surface by pulling upon its two ends the normal pressure upon the surface will vary inversely as the radius of curvature of the surface, the curvature being taken in the plane of the string at that point.

2. If ρ be constant p will be constant; hence, if a circular cylinder be immersed in a fluid, its axis being vertical, the normal pressure on a horizontal arc being uniform throughout its circumference, the compression in the arc will also be constant.

h. THE LAW OF LOADING ON A NORMALLY PRESSED ARC BEING GIVEN, REQUIRED THE EQUATION OF THE ARC

32. *The ties of a suspension bridge being normal to the curve of the cable, and the load uniform along the span, required the equation of the curve of the cable.*

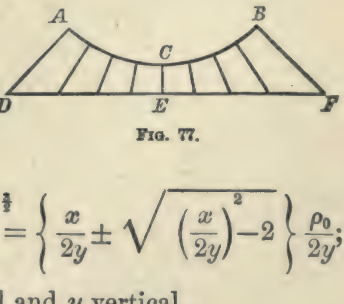


FIG. 77.

$$Ans. \left\{1 + \left(\frac{x}{2y} \pm \sqrt{\left(\frac{x}{2y}\right)^2 - 2}\right\}^{\frac{3}{2}} = \left\{\frac{x}{2y} \pm \sqrt{\left(\frac{x}{2y}\right)^2 - 2}\right\}\frac{\rho_0}{2y},$$

the origin being at C, x horizontal and y vertical.

If $\tan i = \frac{dy}{dx}$, and $\rho_0 =$ the radius of curvature at the vertex, then

$$x = \tfrac{1}{2}\rho_0(1 + \cos^2 i)\sin i,$$
$$y = \tfrac{1}{2}\rho_0 \sin^2 i \cos i.$$

(See solution by Prof. S. W. Robinson, *Journal of the Franklin Institute*, 1863, vol. 46, p. 145; and its application to bridges and arches, vol. 47, p. 152 and p. 361.)

33. *A perfectly flexible, inextensible trough of indefinite length is filled with a fluid, the edges of the trough being parallel and supported in a horizontal plane; required the equation of a cross section.*

The length is assumed to be indefinitely long, so as to eliminate the effect of the *end pieces.* The pressure of a fluid against a surface is always normal to the surface, and varies directly with the depth of the fluid. The actual pressure equals the weight of a prism of water whose base equals the surface pressed, and whose height equals the depth of the centre of gravity of the said surface below the surface of the fluid. The problem may therefore be stated as follows:—*Required the equation of the curve assumed by a cord fixed at two points in the same horizontal, and pressed normally by forces which vary as the vertical distance of the point of application below the said horizontal.*

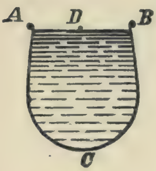


FIG. 78.

Let A and B be the fixed points. Take the origin of the coördinates at D, midway between A and B, and y positive downwards. Let δ be the weight of a unit of volume; then $p = \delta y$, which in equation (o) gives $t = \delta y \rho$, and for the lowest point $t = \delta D \rho_0$; in which D is the depth of the lowest point and ρ_0 the radius of curvature at that point;

$$\therefore \delta y \rho = \delta D \rho_0, \text{ or } \frac{y}{D\rho_0} = \frac{1}{\rho}.$$

But from the Theory of Curves we have

$$\frac{1}{\rho} = \left(1 + \frac{dy^2}{dx^2}\right)^{-\frac{3}{2}} \frac{d^2y}{dx^2};$$

which substituted above, and both sides multiplied by dy, may be put under the form

$$-\tfrac{1}{2}\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{-\frac{3}{2}} d\left(\frac{dy^2}{dx^2}\right) = \frac{y\,dy}{D\rho_0};$$

the integral of which is

$$\left(1 + \frac{dy^2}{dx^2}\right)^{-\frac{1}{2}} = \frac{y^2}{2D\rho_0} + C.$$

But $\dfrac{dy}{dx} = 0$, for $y = D$; $\therefore\ C = \dfrac{2\rho_0 - D}{2\rho_0}$; which substituted and the equation reduced gives

$$(dx^2 + dy^2)^{\frac{1}{2}} = ds = \frac{2\rho_0 D dy}{\sqrt{[4D^2\rho_0^{\,2} - (2D\rho_0 - D^2 + y^2)^2]}}.$$

Squaring and reducing, gives

$$dx = \frac{2\rho_0 D - D^2 + y^2}{\sqrt{[4D^2\rho_0^{\,2} - (2D\rho_0 - D^2 + y^2)^2]}}\, dy.$$

These may be integrated by means of *Elliptic Functions.* Making $y = D\cos\phi$, and $c = \dfrac{D}{4\rho_0}$, they may be reduced to known forms. Using *Legendre's* notation, we have

$$s = \sqrt{\rho_0 D}\, \mathrm{F}_{(c,\,\phi)};$$

$$x = 2\sqrt{\rho_0 D}\left\{-\tfrac{1}{2}\,\mathrm{F}_{(c,\,\phi)} + \mathrm{E}_{(c,\,\phi)}\right\}.$$

(See Article by the Author in the *Journal of the Franklin Institute,* 1864, vol. 47, p. 289.)

CHAPTER V.

RELATION BETWEEN THE INTENSITIES OF FORCES ON DIFFERENT PLANES WHICH CUT AN ELEMENT.

87. Distributed Forces are those whose points of application are distributed over a surface or throughout a mass. The attraction of one mass for another is an example of the latter, some of the properties of which have been discussed in the Chapter on Parallel Forces; similarly, when one part of a body is subjected to a pull or push, the forces are transmitted through the body to some other part, and are there resisted by other forces. If the body be intersected by a plane, the forces which pass through it will be distributed over its surface. *Planes having different inclinations being passed through an element, it is proposed to find the relation between the intensities of the forces on the different planes.*

88. Definitions. *Stresses* are forces distributed over a surface. In the previous chapters we have *assumed* that forces are applied at points, but in practice they are always *distributed.*

A strain is the distortion of a body caused by a stress. Stresses tend to change the form or the dimensions of a body. Thus, a *pull* elongates, a *push* compresses, a *twist* produces torsion, etc. (See *Resistance of Materials.*)

A simple stress is a pull or thrust. Stresses may be compound, as a combination of a twist and a pull.

A direct *simple* stress is a pull or thrust which is normal to the plane on which it acts.

A *pull* is considered *positive*, and a *push*, *negative.*

The *intensity of a stress* is the force on a unit of area, if it be constant; but, if it be *variable*, it is the ratio of the stress on an elementary area to the area.

To form a clear conception of the forces to which an element is subjected, conceive it to be removed from the body and then

subjected to such forces as will produce the same strain that it had while in the body.

89. FORMULAS FOR THE INTENSITY OF A STRESS. Let F be a direct simple stress acting on a surface whose area is A, and p the measure of the intensity, then

$$p = \frac{F}{A}, \tag{92}$$

when the stress is uniform, and

$$p = \frac{dF}{dA}, \text{ when it is variable.}$$

If the stress be variable we will assume that the section is so small that the stress may be considered uniform over its surface.

90. DIRECT STRESS RESOLVED. Let the prismatic element AB, Fig. 79, be cut by an oblique plane DE. Let the stress F be simple and direct on the surface CB, and

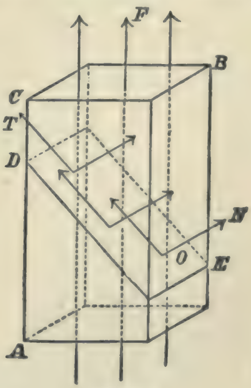


FIG. 79.

$N =$ the normal component of F on DE;

$T =$ the component of F along the plane DE, which is called the *tangential component*;

$\theta = FON =$ the angle between the action-line of the force and a normal to the plane DE, and is called the *obliquity of the plane*;

$A =$ the area of CB, and A' that of DE.

Then, according to equations (62), we have

$$N = F \cos \theta;$$
$$T = F \sin \theta.$$

From the figure we have

$$A' = A \sec \theta,$$

hence, on the plane DE, we have

$$\left.\begin{aligned} \textit{Normal intensity}, p_n &= \frac{N}{A'} = \frac{F\cos\theta}{A\sec\theta} = p\ \cos^2\theta; \\ \textit{Tangential intensity}, p_t &= \frac{T}{A'} = \frac{F\sin\theta}{A\sec\theta} = p\sin\theta\cos\theta. \end{aligned}\right\} \quad (93)$$

Pass another plane perpendicular to DE, having an obliquity of $90° - \theta$; then, accenting the letters, we have

$$\left.\begin{aligned} p'_n &= p\sin^2\theta; \\ p'_t &= p\cos\theta\sin\theta. \end{aligned}\right\} \quad (94)$$

This result is the same as if a direct stress acting upon a plane perpendicular to CB, having an obliquity of $90° - \theta$ in reference to DE, be resolved normally and tangentially to the latter.

Combining equations (93) and (94) we readily find

$$\left.\begin{aligned} p_n + p'_n &= p; \\ p_t &= p'_t; \end{aligned}\right\} \quad (95)$$

that is, *when an element (or body) under a direct simple stress is intersected by two planes the sum of whose obliquities is* 90 *degrees, the sum of the intensities of the normal components of the stress equals the intensity of the direct simple stress, and the intensities of the tangential stresses are equal to each other.*

91. SHEARING STRESS. The tangential stress is commonly called a *shearing stress.* It tends to draw a body sidewise along its plane of action, or along another plane parallel to its plane of action. Its action may be illustrated as follows:— Suppose that a pile composed of thin sheets or horizontal layers of paper, boards, iron, slate, or other substance, having friction between the several layers, be acted upon by a horizontal force applied at the top of the pile, tending to move it sidewise. It will tend to draw each layer upon the one immediately beneath it, and the total force exerted between each layer will equal the applied force, and the resistance at the bottom of the pile will be equal and opposite to that of the applied force. If other horizontal forces are applied at different points along the vertical face of the pile, the total tangential force at the base of the pile will equal the algebraic sum of all the applied forces.

A shearing stress and the resisting force constitute a couple,

and as a single couple cannot exist alone, so a pair of shearing stresses necessitate another pair for equilibrium

When the direct simple stresses on the faces of a rectangular parallelopipedon are of equal intensity, the shearing stresses will be of equal intensity.

Let Fig. 80 represent a paralellopipedon with direct and shearing stresses applied to its several faces. At present suppose that all the forces are parallel to the plane of one of the faces, as *abfe*, and call it *a plane of the forces;* then will the *planes of action*, which, in this case, will be four of the faces of the parallelopipedon, be perpendicular to *a plane of the forces.*

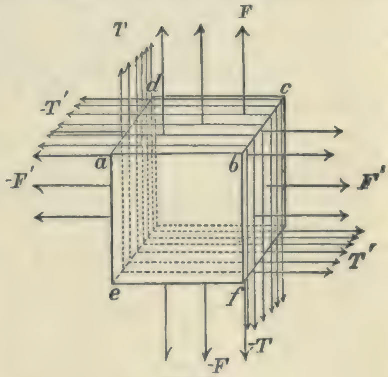


FIG. 80.

If the direct stress $+F = -F$, and $+F' = -F'$, they will equilibrate each other. The moment of the tangential force T, will be

$$p_t \times \textit{area } fc \times ab;$$

and of T'

$$p'_t \times \textit{area } ac \times bf.$$

The couple $T.ab$ tends to turn the element to the right and $T'.bf$ to the left, hence, for equilibrium, we have

$$p_t \times \textit{area } fc \times ab = p'_t \times \textit{area } ac \times bf;$$

but *area* $fc \times ab =$ *area* $ac \times bf =$ the volume of the element, hence

$$p_t = p'_t. \tag{96}$$

The effect of a pair of shearing stresses is to distort the element, changing a rectangular one into a rhomboid, as shown in Fig. 81.

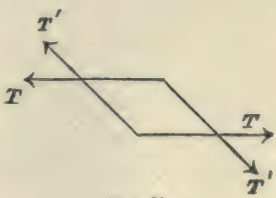


FIG. 81.

Direct stresses are directly opposed to each other in the same plane or on opposite surfaces; shearing stresses act on parallel planes not coincident.

92. NOTATION. A very good notation was devised by Coriolis, which has been used since 1837, and is now commonly employed for the *general* investigations on this subject. It is as follows:—

Let p be a *typical* letter to denote the intensity of a stress of some kind; p_x the intensity of a stress on a plane normal to x; p_{xx} the intensity of a stress on a plane normal to x and in a direction parallel to x, and hence indicates the intensity of a *direct simple stress;* and p_{xy} the intensity of a stress on a plane normal to x but in the direction of y, and hence indicates the intensity of a *shearing stress.* Or, generally, *the first sub-letter indicates a normal to the plane of action and the second one the direction of action.* Hence we have

INTENSITIES OF THE FORCES

parallel to

$$\begin{array}{ccc} x & y & z \\ p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{array}\left.\vphantom{\begin{array}{c}a\\a\\a\end{array}}\right\} \text{ on a plane normal to } \left\{\begin{array}{l} x; \\ y; \\ z. \end{array}\right.$$

If direct stresses only are considered, one sub-letter is sufficient; as p_x, p_y, or p_z.

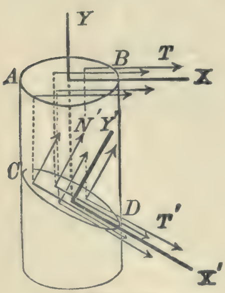


FIG. 82.

93. TANGENTIAL STRESS RESOLVED. Let T be the tangential stress on the right section $AB = A$, the section being normal to y, then

$$p_{yx} = T \div A.$$

Let CD be an oblique section, normal to the axis y'; x' and x being in the plane of the axes y and y'; then will the angle between y and y' be the obliquity of the plane CD. This we will denote by (yy').

Let the tangential force be parallel to the axis of x.

Resolving this force, we have

$$\textit{Normal component on } CD = T \sin (yy');$$
$$\textit{Tangential component on } CD = T \cos (yy').$$

Dividing each of these by *area* $CD = AB \div \cos(yy')$, we have

$$\left.\begin{array}{l} \textit{Normal intensity} = p_{y'y'} = \dfrac{T \sin(yy') \cos(yy')}{area\ AB} = p_{yx} \sin(yy') \cos(yy'); \\ \textit{Tangential intensity} = p_{y'x'} = \dfrac{T \cos^2(yy')}{area\ AB} = p_{yx} \cos^2(yy'); \end{array}\right\} \quad (97)$$

and for a tangential stress on a plane normal to x, resolved upon the same oblique plane CD, we have

$$\left.\begin{array}{l} p'_{y'y'} = p_{xy} \cos(yy') \sin(yy'); \\ p'_{y'x'} = p_{xy} \sin^2(yy'). \end{array}\right\} \quad (98)$$

If the tangential stresses on both planes (one normal to y, and the other normal to x) are alike, and the obliquity of the plane CD less than 90°, the resultant of their tangential components will be the difference of the two components, as given by equations (97) and (98); that is, it will be $p_{y'x'} - p'_{y'x'}$; but the normal intensity will be the sum of the components as given by the same equations. The reverse will be true in regard to the *direct stresses.*

94. *Let a body be subjected to a direct simple stress; it is required to find the stresses on any two planes perpendicular to one another and to the plane of the forces; also the intensity of the stress on a third plane perpendicular to the plane of the forces; and the normal and tangential components on that plane.*

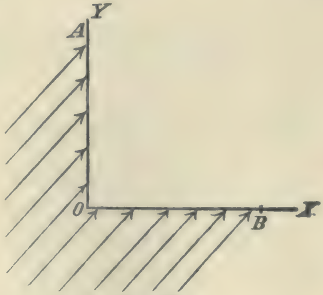


Fig. 83.

Let the forces be parallel to the plane of the paper; AO and OB, planes perpendicular to one another and to the plane of the paper, having any obliquity with the forces. Let the axis of x coincide with OB, and y with AO. Let AB be a third plane, also perpendicular to the plane of the paper, cutting the other planes at any angle. Take y' perpendicular to AB and x' parallel to it and to the plane of the paper.

The oblique forces may be resolved normally and tangentially to the planes AO and OB, by means of equations (93) and (94). The problem will then be changed to that shown in Fig. 84, in which one set of stresses is simple and direct, and the other set tangential; and, according to Article 91, the *intensity* of the shearing stress on the two planes will be the same; hence, for this case

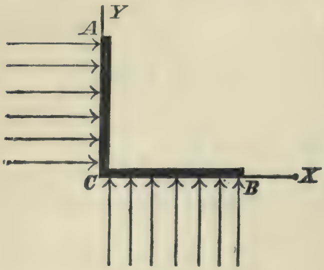


FIG. 84.

$$p_{xy} = p_{yx}.$$

The intensity of the total normal stress on the plane AB will be the sum of the normal components given by equations (93), (94), (97) and (98), and the total tangential stress will be the sum of the components of the tangential stress given by the same equations; hence

$$\left.\begin{array}{l} p_{y'y'} = p_{xx}\sin^2(yy') + p_{yy}\cos^2(yy') + 2p_{xy}\sin(yy')\cos(yy'); \\ p_{y'x'} = \{p_{xx} - p_{yy}\}\sin(yy')\cos(yy') + p_{xy}\{\cos^2(yy') - \sin^2(yy')\}. \end{array}\right\} \quad (99)$$

The *resultant* stress on AB will be, according to equation (46), θ being 90°,

$$p_r = \sqrt{p^2_{y'y'} + p^2_{y'x'}}; \qquad (100)$$

and the inclination of the resultant stress to the normal, y', will be

$$\tan(ry') = \frac{p_{y'x'}}{p_{y'y'}}. \qquad (101)$$

95. DISCUSSION OF EQUATIONS (99).

A. Find the inclination of the plane on which there is no tangential stress.

In the 2d of equations (99) make $p_{y'x'} = 0$, and representing this particular angle by (YY′), we find

$$\tan 2(\text{YY}') = \frac{2\sin(yy')\cos(yy')}{\cos^2(yy') - \sin^2(yy')} = \frac{2p_{xy}}{p_{yy} - p_{xx}}, \qquad (102)$$

which gives two angles differing from each other by 90°, or, the planes will be perpendicular to one another.

Hence, *in every case of a direct simple stress upon a pair of planes perpendicular to one another and to a plane of the stresses, there are two planes, also perpendicular to one another and to the plane of the stresses, on which there is no tangential stress.*

These two directions are called *principal axes of stress.*

Principal axes of stress are the normals to two planes perpendicular to one another on which there is no tangential stress.

Principal stresses are such as are parallel to the principal axes of stress. (In some cases there is a third principal stress perpendicular to the plane of the other two.)

The formulas for the stresses become most simple by referring them directly to the principal axes.

a. Let one of the direct stresses be zero.

Equation 102 gives

$$\tan 2(\text{YY}') = \frac{2p_{xy}}{p_{xx}} \tag{103}$$

b. Let one of the direct stresses be a pull, and the other a push.

Then

$$\tan 2(\text{YY}') = \frac{2p_{xy}}{p_{xx} + p_{yy}} \tag{104}$$

c. Let them act in opposite senses and equal to each other.

Then

$$\tan 2(\text{YY}') = \frac{p_{xy}}{p_{xx}}. \tag{105}$$

d. Let there be no tangential stress on the original planes, or $p_{xy} = 0$.

Then,

$$\tan 2(\text{YY}') = 0; \quad \therefore (\text{YY}') = 0 \text{ or } 90°;$$

and the original planes are *principal planes.*

e. Let there be no direct stresses.

Then,

$$\tan 2(\text{YY}') = \infty; \text{ or } (\text{YY}') = 45^\circ \text{ or } 135^\circ; \qquad (106)$$

that is, *if on two planes, perpendicular to one another and to the plane of the stresses, there are no direct stresses, then will the stress on two planes, perpendicular to one another and to the plane of the stresses, whose inclination with the original planes is* 45°, *be simple and direct.*

f. Let the direct stresses be equal to one another and act in the same sense, and let there be no shearing on the original planes.

Then

$$\tan 2(\text{YY}') = \frac{0}{0};$$

and (YY') is indeterminate; hence, in this case every plane perpendicular to a plane of the stress will be a principal plane

Examples.

1. A rough cube, whose weight is 550 pounds, rests on a horizontal plane. A stress of 150 pounds applied at the upper face pulls vertically upward, and another direct stress of 125 pounds, applied at one of the lateral faces, tends to draw it to the right, while another direct stress of 50 pounds tends to draw it to the left; required the position of the planes on which there are no tangential stresses.

If the cube is of finite size it will be necessary to modify the problem, in order to make it agree with the hypothesis under which the formulas have been established. The force of gravity being distributed throughout the mass, would cause a variable stress, and the surface of no shear would be curved instead of plane. We will therefore assume *that the cube is without weight, and the* 550 *pounds is applied directly to the lower surface.* Then the vertical stress will be 150 pounds, the remaining 400 pounds being resisted directly by the plane on which it rests, and so far as the present problem is concerned, only produces friction for resisting the shearing stress. The direct horizontal

stress will be 50 pounds, the remaining 75 pounds producing a shearing on the horizontal plane. The former force tends to turn the cube right-handed by rotating it about the lower right-hand corner, thus producing a reaction or vertical tangential stress of 75 pounds. Let the area of each face of the cube be unity, then we have

$$p_{xy} = 75 \text{ pounds}; \quad p_{xx} = 50 \text{ pounds}; \quad p_{yy} = 150 \text{ pounds};$$

and these in (102) give

$$\tan 2(\text{YY}') = \frac{2 \times 75}{50 - 150} = 1.5;$$

$$\therefore (\text{YY}') = +\,28°\ 9'\ 18'', \text{ or } -\ 61°\ 50'\ 42''.$$

If the body be divided along either of these planes, the forces will tend to lift one part directly from the other without producing sliding upon the plane of division.

2. A rough body, whose weight is 100 pounds, rests on an inclined plane; required the normal and tangential components on the plane. (Use Eq. (93).)

3. A block without weight is *secured* to a horizontal plane and thrust downward by a stress whose intensity is 150 pounds, and pulled towards the right by a stress whose intensity is 150 pounds, and to the left with an intensity of 100 pounds; required the plane of no shear.

4. A cube rests on a horizontal plane, and one of its vertical faces is forced against a vertical plane by a stress of 200 pounds applied at the opposite face, and on one of the other vertical faces is a direct pulling stress of 75 pounds, which is directly opposed by a stress of 50 pounds on the opposite vertical face; required the position of the plane of no shear.

In this case the weight of the cube would be a third principal stress, but it is eliminated by the conditions of the problem. The shearing stress is 25 pounds; and because the direct stresses are unlike, we use Eq. (104).

5. A rectangular parallelopipedon stands on a horizontal plane, and on the opposite pairs of vertical faces tangential

stresses of equal intensities are applied; required the position of the plane of no shear. (See Eq. (106).)

6. In the preceding problem find the intensity of the direct stress on the plane of no shear. (Substitute the proper quantities in the 1st of (99).)

B. To find the planes of action for maximum and minimum normal stresses, and the values of the stresses.

Equate to zero the first differential coefficient of the 1st of Equations (99), and we have

$$\left.\begin{array}{l} 2p_{xx}\cos(yy')\sin(yy') - 2p_{yy}\sin(yy')\cos(yy') \\ -\,2p_{xy}\sin^2(yy') + 2p_{xy}\cos^2(yy') = 0; \end{array}\right\} \quad (107)$$

$$\therefore\ \tan 2(yy') = \frac{-2p_{xy}}{p_{xx} - p_{yy}};$$

which, being the same as (102), shows that on those planes which have no shearing stress, the direct stress will be either a maximum or a minimum. Testing this value by the second differential coefficient, we find that one of the values of (YY') gives a maximum and the other a minimum.

Comparing (107) with the second of (99), shows that *the first differential coefficient of the value of the direct stress on any plane is twice the shearing stress on that plane.*

From (107), observing that $\cos(yy') = \sqrt{1 - \sin^2(yy')}$, we find

$$\sin^2(\text{YY}') = \frac{1}{2}\left\{1 \mp \frac{p_{yy} - p_{xx}}{\sqrt{(p_{yy} - p_{xx})^2 + 4p^2_{xy}}}\right\}; \quad (108)$$

and these values in the 1st of (99), and the maximum and minimum values designated by $p_{\text{Y}'}$, give

$$p_{\text{Y}'} = \tfrac{1}{2}(p_{xx} + p_{yy}) \pm \sqrt{\{\tfrac{1}{4}(p_{xx} - p_{yy})^2 + p^2_{xy}\}}; \quad (109)$$

in which the upper sign gives the *maximum*, and the lower the *minimum* stress. These are ***principal stresses***, and we denote them by one sub-letter.

a. *If* $p_{xy} = 0$, we have

$$(yy') = 0° \text{ or } 90°, \text{ as we should.}$$

b. *If* $p_{yy} = 0$, we have

$$\left.\begin{aligned} maximum, p_{Y'} &= \tfrac{1}{2}p_{xx} + \sqrt{\tfrac{1}{4}p^2_{xx} + p^2_{xy}}\,; \\ minimum, p_{Y'} &= \tfrac{1}{2}p_{xx} - \sqrt{\tfrac{1}{4}p^2_{xx} + p^2_{xy}}\,; \end{aligned}\right\} \quad (110)$$

hence, the maximum normal stress will be of the same *kind* as the principal direct stress, p_{xx}; that is, if the latter is a *pull*, the former will also be a *pull*, and the minimum principal stress will be of the opposite kind.

c. If there are no direct stresses p_{xx} will also be zero, and we have

$$(\text{YY}') = 45° \text{ or } 135°;$$

and

$$maximum\ p_{Y'} = p_{xy} = -p_{Y'} \text{ for } minimum\,;$$

that is, the principal stresses will have the same intensity as the shearing stresses, and act on planes perpendicular to one another, and inclined 45° to the original planes.

EXAMPLES.

1. Suppose that a rectangular box rests on one end, and that one pair of opposite vertical sides press upon the contents of the box with an intensity of 20 pounds, and the other pair of vertical faces press with an intensity of 40 pounds, and that horizontal tangential stresses, whose intensities are 10 pounds, are applied to the vertical faces, one pair tending to turn it to the right, and the other to the left; required the position of the vertical planes of no shearing, and the maximum and minimum values of the direct stresses.

2. For an application of Equations (103) and (110) to the stresses in a beam, see the Author's *Resistance of Materials*, 2d edition, pp. 236–240.

C. To find the position of the planes of maximum and minimum shearing.

Equate to zero the first differential coefficient of the second of (99) and reduce, denoting the angles sought by (YY'), and we find,

$$-\cot 2(YY') = \tan 2(\text{yy}');$$

$$\therefore 2YY' = 2(\text{yy}') + 90°;$$

or,

$$YY' = \text{yy}' + 45°;$$

that is, the planes of maximum and minimum shear make angles of 45 *degrees with the* PRINCIPAL PLANES.

D. Let the planes be PRINCIPAL SECTIONS.

Then the stresses will be *principal stresses*, and $p_{xy} = 0$. Using a single subscript for the direct stresses, equations (99) become

$$\left.\begin{aligned} p_{y'} &= p_x \sin^2 (yy') + p_y \cos^2 (yy'); \\ p_{y'x'} &= (p_x - p_y) \sin (yy') \cos (yy'). \end{aligned}\right\} \qquad 111)$$

a. Let $p_x = p_y$, then

$$p_{y'} = p_x; \text{ and } p_{y'x'} = 0;$$

that is, *when two principal stresses are alike and equal on a pair of planes perpendicular to the plane of the stresses, the normal intensity on every plane perpendicular to the plane of the stresses will be equal to that on the principal planes, and there will be no shearing on any plane.*

This condition is realized in a perfect fluid, and hence very nearly so in gases and liquids, since they offer only a very slight resistance to a tangential stress. If a vessel of any liquid be intersected by two vertical planes perpendicular to one another, the pressure per square inch will be the same on both, and will be normal to the planes; hence, according to the above, it will be the same upon all planes traversing the same point. This is only another way of stating the fact that fluids press equally in all directions.

b. To find the planes on which there will be no normal pressure.

For this $p_{y'}$ in (111) will be zero;

$$\therefore \tan (yy') = \sqrt{\frac{p_x}{p_y}} \sqrt{-1};$$

which, being imaginary, shows that it is impossible when the stresses are alike; but if they are *unlike*, we have

$$\tan (yy') = \sqrt{\frac{-p_x}{p_y}} \sqrt{-1} = \sqrt{\frac{p_x}{p_y}}.$$

If $p_x = -p_y$, then $(yy') = 45°$, and the 2d of (111) gives

$$p_{y'x'} = p_x;$$

which shows that when the direct stresses are *unlike* and of *equal intensity* on planes perpendicular to one another, the shearing stress on a plane cutting both the others at an angle of 45 degrees, will be of the same intensity.

Let $(yy') = 45°$, or 135°, then (111) become

$$\left.\begin{aligned} p_{y'} &= \tfrac{1}{2}(p_x + p_y); \\ p_{y'x'} &= \pm \tfrac{1}{2}(p_x - p_y); \end{aligned}\right\} \qquad (112)$$

in the latter of which the upper sign gives a maximum, and the lower a minimum value.

Using the upper sign, we find

$$\left.\begin{aligned} p_x &= p_{y'} + p_{y'x'}; \\ p_y &= p_{y'} - p_{y'x'}. \end{aligned}\right\} \qquad (113)$$

96. Problem. *Find the plane on which the obliquity of the stress is greatest, the intensity of that stress, and the angle of its obliquity.*

Let the stresses be *principal stresses* and of the *same kind*, and ϕ the angle of obliquity of the required plane to the stress; then

$\sin \phi = \dfrac{p_x - p_y}{p_x + p_y}$; *the intensity* $= \sqrt{(p_x p_y)}$; and the angle between the principal plane x and the required plane $= 45° - \frac{1}{2}\phi$.

If the principal stresses are *unlike*, then

$\sin \phi' = \frac{p_x + p_y}{p_x - p_y}$; *the intensity* $= \sqrt{-p_x p_y}$, and the angle between the principal plane x, and the oblique plane $= 45° - \frac{1}{2}\phi'$.

Example.

If a body of sand is retained by a vertical wall and the *intensity* of the horizontal push is 25 pounds, and of the vertical pressure is 75 pounds; required the plane on which the resultant has the greatest obliquity, and the intensity of the stress on that plane.

CONJUGATE STRESSES.

97. A pair of stresses, each acting parallel to the plane of action of the other, and whose action-lines are parallel to a plane which is perpendicular to the line of intersection of the planes of action, are called *conjugate stresses.*

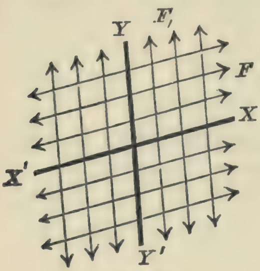


Fig. 85.

Thus, in Fig. 85, one set of stresses acts on the plane YY, parallel to the plane XX, and the other set on XX, parallel to YY. In a rigid body the intensities of these sets of stresses are independent of each other; for each set equilibrates itself. *Principal stresses* are also *conjugate.*

There may be *three conjugate stresses* in a body, and *only three.* For, in Fig. 85, there may be a third stress on the plane of the paper, which may be parallel to the line of intersection of the planes XX and YY, and each stress will be parallel to the plane of the other two. A fourth stress cannot be introduced which will be conjugate to the other three.

Conjugate stresses may be resolved into normal and tangential components on their planes of action, and treated according to the preceding articles. The fact that the stresses have the same obliquity, being the complement of the angle made by the planes, simplifies some of the more general problems of stresses.

GENERAL PROBLEM.

98. *Given the stresses on the three rectangular coördinate planes; required the stresses on any oblique plane in any required direction.*

As before, the element is supposed to be indefinitely small. Let *abc* be the oblique plane, the normal to which designate by n. The projection of a unit of area of this plane on each of the coördinate plains, gives respectively

$$\cos(nx), \cos(ny), \cos(nz).$$

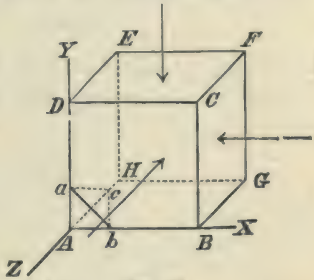


FIG. 86.

The direct stress parallel to x acting on the area $\cos(nx)$ will give a stress of $p_{xx}\cos(nx)$, and the tangential stress normal to y and parallel to x will produce a stress $p_{yx}\cos(ny)$, and similarly the tangential stress normal to z and parallel to x gives $p_{zx}\cos(nz)$; hence the total stress on the unit normal to n and parallel to x will be

$$p_{nx} = p_{xx}\cos(nx) + p_{yx}\cos(ny) + p_{zx}\cos(nz);$$

similarly,

$$\left.\begin{aligned} p_{ny} &= p_{xy}\cos(nx) + p_{yy}\cos(ny) + p_{zy}\cos(nz);\\ p_{nz} &= p_{xz}\cos(nx) + p_{yz}\cos(ny) + p_{zz}\cos(nz). \end{aligned}\right\} \quad (114)$$

Let these be resolved in any arbitrary direction parallel to s. To do this multiply the first of the preceding equations by $\cos(sx)$, the second by $\cos(sy)$, and the third by $\cos(sz)$, and add the results.

For the purpose of abridging the formulas, let $\cos(nx)$ be written Cnx, and similarly for the others. Then we have

$$\left.\begin{aligned} p_{ns} = p_{xx}CnxCsx &+ p_{yy}CnyCsy + p_{zz}CnzCsz \\ &+ p_{yz}(CnyCsz + CnzCsy) + p_{zx}(CnzCsx \\ &+ CnxCsz) + p_{xy}(CnxCsy + CnyCsx). \end{aligned}\right\} \quad (115)$$

This expression being *typical*, we substitute x' for n and s, and thus obtain an expression for the intensity on a surface normal to x' and parallel to x'. Or generally, substitute successively x', y', z' for n and s, and we obtain the following formulas:

DIRECT STRESSES.

$$p_{x'x'} = p_{xx}C^2xx' + p_{yy}C^2yx' + p_{zz}C^2zx' + 2p_{yz}Cyx'Czx' + 2p_{zx}Czx'Cxx' + 2p_{xy}Cxx'Cyx';$$

$$p_{y'y'} = p_{xx}C^2xy' + p_{yy}C^2yy' + p_{zz}C^2zy' + 2p_{yz}Cyy'Czy' + 2p_{zx}Czy'Cxy' + 2p_{xy}Cxy'Cyy';$$

$$p_{z'z'} = p_{xx}C^2xz' + p_{yy}C^2yz' + p_{zz}C^2zz' + 2p_{yz}Cyz'Czz' + 2p_{zx}Czz'Cxz' + 2p_{xy}Cxz'Cyz';$$

TANGENTIAL STRESSES.

$$p_{y'z'} = p_{xx}Cxy'Cxz' + p_{yy}Cyy'Cyz' + p_{zz}Czy'Czz' + p_{yz}(Cyy'Czz' + Cyz'Czy') + p_{zx}(Czy'Cxz' + Czz'Cxy') + p_{xy}(Cxy'Cyz' + Cxz'Cyy');$$

$$p_{z'x'} = p_{xx}Cxz'Cxx' + p_{yy}Cyz'Cyx' + p_{zz}Czz'Czx' + p_{yz}(Cyz'Czx' + Czz'Cyx') + p_{zx}(Czz'Cxx' + Czx'Cxz') + p_{xy}(Cxz'Cyx' + Cxx'Cyz');$$

$$p_{x'y'} = p_{xx}Cxx'Cxy' + p_{yy}Cyx'Cyy' + p_{zz}Czx'Czy' + p_{yz}(Cyx'Czy' + Cyy'Czx') + p_{zx}(Czx'Cxy' + Czy'Cxx') + p_{xy}(Cxx'Cyy' + Cxy'Cyx').$$

It may be shown *that for every state of stress in a body there are three planes perpendicular to each other, on which the stress is entirely normal.*

[These equations are useful in discussing the general *Theory of the Elasticity of Bodies.*]

These formulas apply to *oblique* axes as well as *right*, only it should be observed when they are oblique that $p_{y'z'}$ is not a stress on a plane normal to y', parallel to z', but on a plane normal to x' resolved in the proper direction.

CHAPTER VI.

VIRTUAL VELOCITIES.

99. DEF. If the point of application of a force be moved in the most arbitrary manner an indefinitely small amount, the projection of the path thus described on the original action-line of the force is called *a virtual velocity*. The product of the force into the virtual velocity is called *the virtual moment*. Thus, in Fig. 87, if a be the point of application of the force F, and ab the arbitrary displacement, ac will be the virtual velocity, and $F.ac$ the virtual moment.

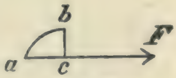


FIG. 87.

The path of the displacement must be so short that it may be considered a straight line; but in some cases its length may be finite.

If the projection falls upon the action-line, as in Fig. 87, the virtual velocity will be considered *positive*, but if on the line prolonged, it will be *negative*.

100. PROP. *If several concurring forces are in equilibrium, the algebraic sum of their virtual moments will be zero.*

Using the notation of Article (47), and in addition thereto let

l be the length of the displacement; and

p, q, and r the angles which it makes with the respective coördinate axes;

then will the projections of l on the axes be

$$l \cos p, \qquad l \cos q, \qquad l \cos r,$$

respectively. Multiplying equations (50), by these respectively, we have

$$F_1 \cos \alpha_1 \, l \cos p + F_2 \cos \alpha_2 \, l \cos p + \text{etc.} = 0;$$
$$F_1 \cos \beta_1 \, l \cos q + F_2 \cos \beta_2 \, l \cos q + \text{etc.} = 0;$$
$$F \cos \gamma_1 \, l \cos r + F_2 \cos \gamma_2 \, l \cos r + \text{etc.} = 0.$$

Adding these together term by term, observing that

$$\cos a \cos p + \cos \beta \cos q + \cos \gamma \cos r = \cos (Fl);$$

which is the cosine of the angle between the action-line of F and the line l; and that $l \cos (Fl) = \delta f$ (read, variation f) is the virtual velocity of F, we have

$$F_1\delta f_1 + F_2\delta f_2 + F_3\delta f_3 + \text{etc.} = \Sigma F\delta f = 0; \qquad (116)$$

which was to be proved.

101. *If any number of forces in a* SYSTEM *are in equilibrium, the sum of their virtual moments will be zero.*

Conceive that the point of application of each force is connected with all the others by rigid right lines, so that the action of all the forces will be the same as in the actual problem. If any of the lines thus introduced are not subjected to stress, they do not form an essential part of the system and may be cancelled at first, or considered as not having been introduced. Let the system receive a displacement of the most arbitrary kind. At each

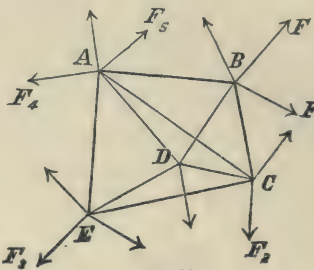


FIG. 88.

point of application of a force or forces, the stresses in the rigid lines which meet at that point, combined with the applied force or forces at the same point, are necessarily in equilibrium, and by separating it from the rest of the system, we have a system of concurrent forces. Hence, for the point B, for instance, we have, according to (116),

$$F\delta f + F_1\delta f_1 + \text{etc.} + BC\delta BC + BA\delta BA + BD\delta BD = 0;$$

in which BC, etc., are used for the tension or compression which may exist in the line. But when the point C is considered, we will have $BC\delta BC$ with a contrary sign from that in the preceding expression, and hence their sum will be zero.

Proceeding in this way, as many equations may be established as there are points of application of the forces; and adding the equations together, observing that all the expressions which represent stresses on the lines disappear, we finally have

$$\Sigma F\delta f = 0. \qquad (117)$$

The converse is evidently true, that when *the sum of the virtual moments is zero the system is in equilibrium.*

Equations (116) and (117) are no more than the *vanishing equations for work.* If a *system* of forces is in equilibrium it does no work. This principle is easily extended to Dynamics. For, the work which is stored in a moving body equals that done by the impelling force above that which it constantly does in overcoming resistances. Thus, when friction is overcome, the impelling forces accomplish work in overcoming this resistance, and all above *that* is stored in the moving mass. Letting R be the resultant of all the impressed forces producing motion, and s the path described by the body, we have

$$R\delta r - \Sigma m \frac{d^2s}{dt^2}\delta s = 0. \tag{118}$$

This is the most general principle of Mechanics, and M. Lagrange made it the fundamental principle of his celebrated work on *Mécanique Analytique*, which consisted chiefly of a discussion of equation (118).

Examples.

1. Determine the conditions of equilibrium of the straight lever.

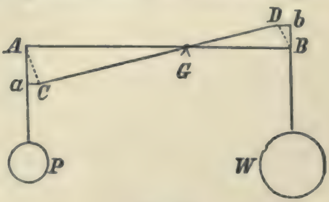


Fig. 89.

Let AB be the lever, having a weight P at one end and W at the other, in equilibrium on the fulcrum G.

Conceive the lever to be turned infinitesimally about G, taking the position CD, then will Aa, which is the projection of the path AC on the action-line of P, be the virtual velocity of P; and similarly Bb will be the virtual velocity of W. The former will be positive and the latter negative; hence

$$P.Aa - W.Bb = 0.$$

The triangles AaC and ACG at the limit are similar, having the right angles AaC and ACG, $aAC = AGC$, and the remaining angle equal. Similarly, bDB is similar to BGD.

$$\therefore \quad \frac{Aa}{Bb} = \frac{CG}{DG} = \text{(at the limit)} \ \frac{AG}{BG};$$

which, substituted in the preceding expression, gives

$$P.AG = W.BG;$$

that is, *the weights are inversely proportioned to the arms.*

If the lever be turned about the end A, we would find in a similar manner that $(P + W).AG = W.AB$; in which $P + W$ is the reaction sustained by the fulcrum G.

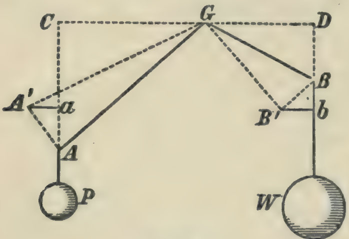


Fig. 90.

2. Find the conditions of equilibrium of the bent lever.

Let AG and GB be the arms of the lever and G the fulcrum. Let it be turned slightly about G; then will Aa and Bb be the respective virtual velocities of P and W;

$$\therefore - P.Aa + W.bB = 0.$$

From G draw GC perpendicular to PA, then will the triangle ACG be similar to AaA', having the angle $AaA' = ACG$; and $aAA' = CGA$. Similarly, the triangle BDG is similar to BbB';

$$\therefore \quad \frac{Aa}{Bb} = \frac{GC}{GD};$$

which, combined with the preceding equation, gives

$$P.GC = W.GD;$$

that is, *the weights are inversely proportional to their horizontal distances from the fulcrum.*

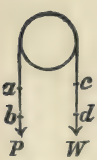


Fig. 91.

3. Find the conditions of equilibrium of the single pulley.

In Fig. 91, let the weight P be moved a distance equal to ab, then will W be moved a distance $cd = ab$; hence, we have

$$- P.ab + W.cd = 0; \quad \therefore P = W.$$

4. On the inclined plane AC, a weight P is held by a force W acting parallel to the plane; required the relation between P and W.

$de = ab$ will be the virtual velocity of W, and ac that of P; and we have

$$- P.ac + W.ab = 0.$$

From the similar triangles abc and ABC, we have

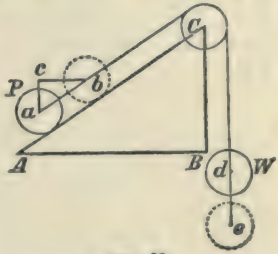


FIG. 92.

$$\frac{ac}{ab} = \frac{CB}{AC} \quad \therefore P.CB = W.AC; \text{ or}$$

$$P : W :: AC : CB.$$

5. On the inclined plane, if the weight P is held by a force W, acting horizontally, required the relation between P and W.

The movement being made, cd will be the virtual velocity of W, which at the limit equals ae, and be will be the virtual velocity of P, and we have

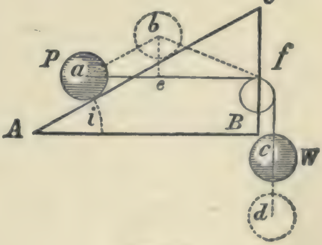


FIG. 93.

$$- P.be + W.ae = 0; \text{ and } ae : eb :: AB : BC,$$

$$\therefore P.CB = W.BA;$$

or, *the weight is to the horizontal force as the base of the triangle is to its altitude*

6. In Fig. 27 show that $Pdr = Wdy$.

7. One end of a beam rests on a horizontal plane, and the other on an inclined plane; required the horizontal pressure against the inclined plane.

This involves the principle of the wedge; for the block ABC may represent one-half of a wedge being forced against the resistance W. Conceive the plane to be moved a distance AA', and that the beam turns

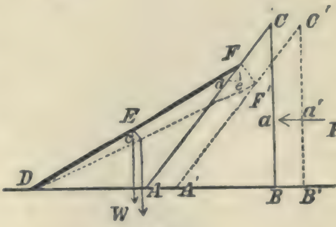


FIG. 94.

about the end D, but is prevented from sliding on the plane; then will the virtual velocity of the horizontal pressure be AA', and that of the weight will be Ec; hence, for equilibrium we have

$$W.Ec - P.AA' = 0. \qquad (a)$$

We now find the relation between Ec and AA'.

Let $l = DF$, the length of the beam;

$a = DE$, the distance from D to the centre of gravity of the beam;

$\alpha = CAB$; $\beta = ADF$.

The end at F will describe an arc FF' about D as a centre. From F' draw $F'd$ parallel to AA', and from F drop a perpendicular Fe upon dF'. Then, from similar triangles, we have

$$Fe = \frac{l}{a} Ec,$$

FF' will be perpendicular to DF, and Fe perpendicular to dF', hence

$$eFF' = ADF = \beta; \; dFe = 90° - \alpha;$$

$$\therefore \; dFF' = 90° - \alpha + \beta;$$

and

$$FF' = Fe \sec \beta = \frac{l}{a} Ec \sec \beta.$$

The triangle FdF' gives

$$\frac{FF'}{AA' = dF'} = \frac{\sin \alpha}{\sin (90° - \alpha + \beta)},$$

hence,

$$\frac{Ec}{AA'} = \frac{a}{l} \; \frac{\sin \alpha \cos \beta}{\cos (\alpha - \beta)};$$

which, substituted in equation (a) above, gives

$$P = W \frac{a}{l} \; \frac{\sin \alpha \cos \beta}{\cos (\alpha - \beta)}.$$

8. Deduce the formula for the triangle of forces from the principle of Virtual Velocities.

CHAPTER VII.

MOMENT OF INERTIA.

(This chapter may be omitted until its principles are needed hereafter (see Ch. X.) Although the expression given below, called the *Moment of Inertia*, comes directly from the solution of certain mechanical problems, yet its principles may be discussed without involving the idea of *force*, the same as any other mathematical expression. The *term* probably originated from the idea that inertia was considered *a force*, and in most mechanical problems which give rise to the *expression* the moment of a force is involved. But the expression is not in the *form* of a simple moment. If we consider a moment as the product of a quantity by an arm, it is of the *form of a moment of a moment*. Thus, dA being the quantity, ydA would be a moment, then considering this as a new quantity, multiplying it by y gives y^2dA, which would be a moment of the moment. Since we do not consider inertia as a force, and since all these problems may be reduced to the consideration of geometrical magnitudes, it appears that some other term might be more appropriate. It being, however, universally used, a change is undesirable unless a new and better one be *universally* adopted.)

DEFINITIONS.

102. The expression, $\int y^2dA$, in which dA represents an element of a body, and y its ordinate from an axis, occurs frequently in the analysis of a certain class of problems, and hence it has been found convenient to give it a special name. It is called *the moment of inertia.*

THE MOMENT OF INERTIA OF A BODY

is the sum of the products obtained by multiplying each element of the body by the square of its distance from an axis.

The axis is any straight line in space from which the ordinate is measured.

The quantity dA may represent an element of a line (straight or curved), a surface (plane or curved), a volume, weight, or mass; and hence the above definition answers for all these quantities.

The moment of inertia of a plane surface, when the axis lies in it, is called a *rectangular moment;* but when the axis is perpendicular to the surface it is called a *polar moment.*

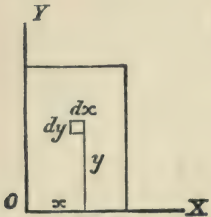


FIG 95.

103. EXAMPLES.

1. Find the moment of inertia of a rectangle in reference to one end as an axis.

Let $b =$ the breadth, and $d =$ the depth of the rectangle. Take the origin of coördinates at O.

We have $dA = dydx;$

and
$$I = \int_0^d \int_0^b y^2 dydx = b \int_0^d y^2 dy = \tfrac{1}{3}bd^3.$$

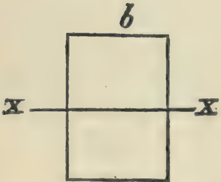


FIG. 96.

2. What is the moment of inertia of a rectangle in reference to an axis through the centre and parallel to one end?

Ans. $\frac{1}{12}bd^3$.

3. What is the moment of inertia of a straight line in reference to an axis through one end and perpendicular to it, the section of the line being considered unity?

Ans. $\frac{1}{3}l^3$.

4. Find the moment of inertia of a circle in reference to an axis through its centre and perpendicular to its surface.

We represent the *polar moment* of inertia by I_p.

Let $r =$ the radius of the circle;
$\rho =$ the radius vector;
$\theta =$ the variable angle; then
$d\rho =$ one side of an elementary rectangle;
$\rho d\theta =$ the other side; and
$dA = \rho d\rho d\theta;$

and, according to the definition, we have

$$I_p = \int_0^r \int_0^{2\pi} \rho^3 d\rho d\theta = \tfrac{1}{2}\pi r^4.$$

5. What is the moment of inertia of a circle in reference to a diameter as an axis? (See Article 105.)

Ans. $\frac{1}{4}\pi r^4$.

FIG. 97.

6. What is the moment of inertia of an ellipse in reference to its major axis; a being its semi-major axis and b, its semi-minor?

Ans. $\frac{1}{4}\pi ab^3$.

7. Find the moment of inertia of a triangle in reference to an axis through its vertex and parallel to its base.

Let b be the base of the triangle, d its altitude, and x any width parallel to the base at a point whose ordinate is y; then $dA = dxdy$, and we have

$$I = \int_0^d \int_0^{\frac{b}{d}y} y^2 dy dx = \frac{b}{d}\int_0^d y^3 dy = \tfrac{1}{4}bd^3.$$

8. What is the moment of inertia of a triangle in reference to an axis passing through its centre and parallel to the base?

Ans. $\frac{1}{36}bd^3$.

9. What is the moment of inertia of an isosceles triangle in reference to its axis of symmetry?

Ans. $\frac{1}{48}b^3d$.

10. Find the moment of inertia of a sphere in reference to a diameter as an axis.

The equation of the sphere will be $x^2 + y^2 + z^2 = R^2$. The moment of inertia of any section perpendicular to the axis of x will be $\frac{1}{2}\pi y^4$; hence for the sphere we have

$$I = \int_{-R}^{+R} \tfrac{1}{2}\pi y^4 dx = \pi \int_0^R (R^2 - x^2)^2\, dx = \tfrac{8}{15}\pi R^5.$$

FORMULA OF REDUCTION.

104. *The moment of inertia of a body, in reference to any axis, equals the moment of inertia in reference to a parallel axis passing through the centre of the body plus the product of the area (or volume or mass) by the square of the distance between the axes.*

This proposition for plane areas was proved in Article 80.

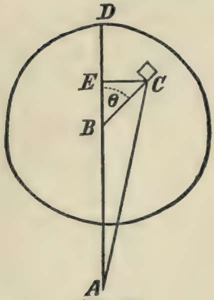


FIG. 98.

To prove it generally, let Fig. 98 represent the projection of a body upon the plane of the paper, B the projection of an axis passing through the centre of the body, A any axis parallel to it, C the projection of any element; $AC = r$, $BC = r_1$, the angle $CBD = \theta$, and $V =$ the volume of the body.

Then

$I_1 = \int r_1^2 dV$ will be the moment of inertia of the volume in reference to the axis through the centre; and

$I = \int r^2 dV$, the moment in reference to the axis through A.

Let $AB = D$, then $AE = D + r_1 \cos\theta$, and $r^2 = r_1^2 \sin^2\theta + (D + r_1 \cos\theta)^2$;

$$\therefore \int r^2 dV = \int r_1^2 dV + 2D\int r_1 \cos\theta dV + D^2 \int dV.$$

But $\int r_1 \cos\theta dV = 0$, since it is the statical moment of the body in reference to a plane perpendicular to AD passing through the centre of the body and perpendicular to the plane of the paper, therefore the preceding equation becomes

$$I = I_1 + VD^2; \qquad (119)$$

which is called the *formula of reduction.*

From this, we have

$$I_1 = I - VD^2. \qquad (120)$$

EXAMPLES.

1. The moment of inertia of a rectangle in reference to one end as an axis being $\frac{1}{3}bd^3$, required the moment in reference to a parallel axis through the centre.

Equation (120) gives

$$I_1 = \tfrac{1}{3}bd^3 - bd\,(\tfrac{1}{2}d)^2 = \tfrac{1}{12}bd^3.$$

2. Given the moment of inertia of a triangle in reference to an axis through its vertex and parallel to the base, to find the moment relative to a parallel axis through its centre.

Example 7 of the preceding Article gives $I = \frac{1}{4}bd^3$; hence equation (120) gives

$$I_1 = \tfrac{1}{4}bd^3 - \tfrac{1}{2}bd\,(\tfrac{2}{3}d)^2 = \tfrac{1}{36}bd^3$$

3. Find the moment of the same triangle in reference to the base as an axis.

Equation (119) gives

$$I = \tfrac{1}{36}bd^3 + \tfrac{1}{2}bd\,(\tfrac{1}{3}d)^2 = \tfrac{1}{12}bd^3.$$

105. To find the relation between the moments of inertia in reference to different pairs of rectangular axes having the same origin.

Let x and y be rectangular axes,
x_1 and y_1, also rectangular, having the same origin;
a = the angle between x and x_1;
I_x = the moment of inertia relatively to the axis x, similarly for
I_y, I_{x_1} and I_{y_1};
$B = \int xydA$; and
$B_1 = \int x_1y_1dA$.

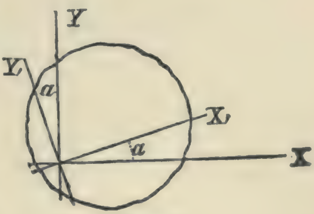


Fig. 90.

For the transformation of coördinates we have

$$x_1 = y\sin a + x\cos a;$$
$$y_1 = y\cos a - x\sin a;$$
$$x_1^2 + y_1^2 = x^2 + y^2.$$

Also

$$dA = dxdy = dx_1dy_1.$$

Hence,

$$\left.\begin{aligned}
I_{x_1} &= \int y_1^2\,dA = I_x\cos^2 a + I_y\sin^2 a - 2B\cos a\sin a;\\
I_{y_1} &= I_x\sin^2 a + I_y\cos^2 a + 2B\cos a\sin a;\\
B_1 &= (I_x - I_y)\cos a\sin a + B\,(\cos^2 a - \sin^2 a);\\
\therefore\ I_{x_1} &+ I_{y_1} = I_x + I_y = I_p;
\end{aligned}\right\}\ (121)$$

the last value of which is found from the expression $\int y_1^2dA + \int x_1^2dA = \int (y_1^2 + x_1^2)\,dA = \int \rho^2dA = I_p$; which shows that the

polar moment equals the sum of two *rectangular moments*, the origin being the same. If the rectangular moments equal one another, we have $I_p = 2I_x$. Thus, in the circle, $I_p = \frac{1}{2}\pi r^4$. (See Ex. 4, Article 103), hence $I_x = \frac{1}{4}\pi r^4$.

The last of equations (121) is an *isotropic* function; since the sum of the moments relatively to a pair of rectangular axes, equals the sum of the moments relatively to any other pair of rectangular axes having the same origin; or, in other words, the sum of the moments of inertia relatively to a pair of rectangular axes, is constant.

To find the maximum or minimum moments we have, from the preceding equations,

$$\frac{dI_{x_1}}{da} = -(I_x - I_y)\cos a \sin a - B(\cos^2 a - \sin^2 a) = 0;$$

and

$$\frac{dI_{y_1}}{da} = +(I_x - I_y)\cos a \sin a + B(\cos^2 a - \sin^2 a) = 0;$$

$$\therefore B_1 = 0.$$

From the first or second of these we have

$$\frac{-2B}{I_x - I_y} = \frac{2\cos a \sin a}{\cos^2 a - \sin^2 a} = \tan 2a.$$

It may be shown by the ordinary tests that when I_{x_1} is a maximum, I_{y_1} will be a minimum, and the reverse; hence *there is always a pair of rectangular axes in reference to one of which the moment of inertia is greater than for any other axis, and for the other it is less.*

These are called *principal axes.*

Thus, in the case of a rectangle, if the axes are parallel to the sides and pass through the centre, we find

$$B = \int\int_{-\frac{1}{2}d}^{+\frac{1}{2}d} xy\,dA = 0;$$

hence x and y are the axes for maximum and minimum moments; and if $d > b$, $\frac{1}{12}bd^3$ is the maximum, and $\frac{1}{12}b^3d$ a minimum moment of inertia for all axes passing through the origin. In a similar way we find that if the origin be at any

other point the axes must be parallel to the sides for maximum and minimum moments.

The preceding analysis gives the position of the axes for maximum and minimum moments, when the moments are known in reference to any pair of rectangular axes. But if the axes for maximum and minimum moments are known as I_x and I_y, then $B = 0$; and calling these $I_{x'}$ and $I_{y'}$, Eqs. (121) become

$$\left.\begin{aligned} I_{x_1} &= I_{x'} \cos^2 a + I_{y'} \sin^2 a; \\ I_{y_1} &= I_{x'} \sin^2 a + I_{y'} \cos^2 a; \\ B_1 &= (I_{x'} - I_{y'}) \cos a \sin a. \end{aligned}\right\} \qquad (122)$$

In the case of a square when the axes pass through the centre $I_x' = I_y'$;

$$\begin{aligned} \therefore\ I_{x_1} &= I_{x'} (\cos^2 a + \sin^2 a) = I_{x'}; \\ I_{y_1} &= I_{y'}, \text{ and} \\ B_1 &= 0; \end{aligned}$$

hence the moment of inertia of a square is the same in reference to all axes passing through its centre. The same is true for all regular polygons, and hence for the circle.

EXAMPLES.

1. To find the moment of inertia of a rectangle in reference to an axis through its centre and inclined at an angle a to one side, we have

$$\begin{aligned} I_x &= \tfrac{1}{12} bd^3 \text{ and } I_y = \tfrac{1}{12} b^3 d \\ \therefore\ I_{x_1} &= \tfrac{1}{12} bd\,(d^2 \cos^2 a + b^2 \sin^2 a); \\ I_{y_1} &= \tfrac{1}{12} bd\,(d^2 \sin^2 a + b^2 \cos^2 a). \end{aligned}$$

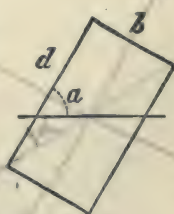


FIG. 100.

2. To find the moment of inertia of an isosceles triangle in reference to an axis through its centre and inclined at an angle a to its axis of symmetry.

We have $I_x = \frac{1}{36} bd^3$ and $I_y = \frac{1}{48} b^3 d$, in which b is the base and d the altitude;

$$\begin{aligned} \therefore\ I_x &= \tfrac{1}{36} bd\,(d^2 \cos^2 a + \tfrac{3}{4} b^2 \sin^2 a) \\ I_y &= \tfrac{1}{36} bd\,(d^2 \sin^2 a + \tfrac{3}{4} b^2 \cos^2 a). \end{aligned}$$

The moment of inertia of a regular polygon about an axis

through its centre may be found by dividing it into triangles having their vertices at the centre of the polygon, and for bases the sides of the polygon; then finding the moments of the triangles about an axis through their centre and parallel to the given axis and reducing them to the given axis by the *formula of reduction.*

If R be the radius of the circumscribed circle, r that of the inscribed circle, and A the area of the polygon; then, for a regular polygon, we would find that

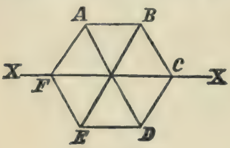


FIG. 101.

$$I = \tfrac{1}{12}A\,(R^2 + 2r^2).$$

For the circle $R = r$,

$$\therefore I = \tfrac{1}{4}\pi r^4,$$

as before found.

For the square, $r = \frac{1}{2}b$, $R = \frac{1}{2}b\sqrt{2}$, and $A = b^2$;

$$\therefore I = \tfrac{1}{12}b^4,$$

as before found.

106. Examples of the moment of inertia of solids.

(The following results are taken from Mosley's *Mechanics and Engineering.*)

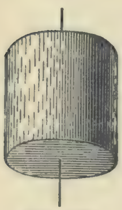


FIG. 102.

1. The moment of inertia of a solid cylinder about its axis of symmetry, r being its radius and h its height, is $\frac{1}{2}\pi h r^4$.

2. If the cylinder is hollow, c the thickness of the solid part and R the mean radius (equal to one-half the sum of the external and internal radii), then $I = 2\pi hcR\,(R^2 + \frac{1}{4}c^2)$.

3. The moment of inertia of a cylinder in reference to an axis passing through its centre and perpendicular to its axis of symmetry is $\frac{1}{4}\pi hr^2\,(r^2 + \frac{1}{3}h^2)$.

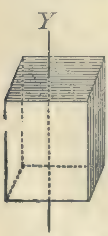


FIG. 103.

4. The moment of inertia of a rectangular parallelopipedon about an axis passing through its centre and parallel to one of its edges. Let a be the length of the edge parallel to the axis, and b and c the lengths of the other edges, then $I = \frac{1}{12}abc\,(b^2 + c^2) = \frac{1}{12}$ *of the volume multiplied by the square of the diagonal of the base.*

5. The moment of inertia of an upright triangular prism having an isosceles triangle for its base, in reference to a vertical axis passing through its centre of gravity.


FIG. 104.

Let the base of the triangle be a, its altitude b, and the altitude of the prism be h, then

$$I = \tfrac{1}{12}abh\,(\tfrac{1}{4}a^2 + \tfrac{1}{3}b^2).$$

6. The moment of inertia of a cone in reference to an axis of symmetry is $\frac{1}{10}\pi r^4 h$. (r being the radius of the base and h the altitude.)

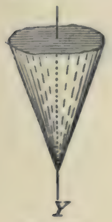


FIG. 105.

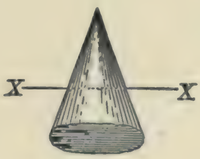


FIG. 106.

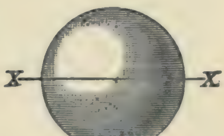


FIG. 107.

7. The moment of inertia of a cone in reference to an axis through its centre and perpendicular to its axis of symmetry is $\frac{1}{20}\pi r^2 h\,(r^2 + \frac{1}{4}h^2)$.

8. The moment of inertia of a sphere about one of its diameters is $\frac{8}{15}\pi R^5$.

9. The moment of inertia of a segment of a sphere about a diameter parallel to the plane of section.

Let R be the radius of the sphere, and b the distance of the plane section from the centre, then

$$I = \tfrac{1}{60}\pi\,(16R^5 + 15R^4b + 10R^2b^3 - 9b^5).$$

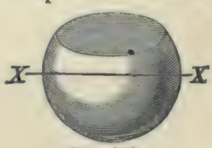


FIG. 108.

RADIUS OF GYRATION.

107. We may conceive the mass to be concentrated at such a point that the moment of inertia in reference to any axis will be the same as for the distributed mass in reference to the same axis.

The radius of gyration is the distance from the moment axis to a point in which, if the entire mass be concentrated, the moment of inertia will be the same as for the distributed mass

The principal radius of gyration is the radius of gyration in reference to a moment axis through the centre of the mass.

Let k = the radius of gyration ;
k_1 = the principal radius of gyration ;
M = the mass of the body ; and
D = the distance between parallel axes ;

then, according to the definitions and equation (119), we have

$$\begin{aligned} Mk^2 &= \Sigma mr^2 \\ &= \Sigma mr_1^2 + MD^2 \\ &= Mk_1^2 + MD^2; \end{aligned}$$

$$\therefore \; k^2 = k_1^2 + D^2; \qquad (123)$$

from which it appears that k is a minimum, for $D = 0$, in which case $k = k_1$; that is, the *principal radius of gyration is the minimum radius for parallel axes.*

We have

$$I_1 = Mk_1^2;$$

$$\therefore \; k_1^2 = \frac{I_1}{M};$$

hence, the square of the principal radius of gyration equals the moment of inertia in reference to a moment axis through the centre of the body divided by the mass.

EXAMPLES.

1. Find the principal radius of gyration of a circle in reference to a rectangular axis.

Example 5 of Article 103 gives, $I_1 = \frac{1}{4}\pi r^4$, which is the moment of an area, hence, we use πr^2 for M, and have

$$k_1^2 = \frac{\frac{1}{4}\pi r^4}{\pi r^2} = \frac{1}{4}r^2.$$

2. For a circle in reference to a polar axis, $k_1^2 = \frac{1}{2}r^2$.

3. For a straight line in reference to a moment axis perpendicular to it, $k_1^2 = \frac{1}{12}l^2$.

4. For a sphere, $k_1^2 = \frac{2}{5}r^2$.

5. For a rectangle whose sides are respectively a and b, in reference to an axis perpendicular to its plane, $k_1^2 = \frac{1}{12}(a^2 + b^2)$.

6. Find the principal radius of gyration of a cone when the moment axis is the axis of symmetry.

CHAPTER VIII.

MOTION OF A PARTICLE FREE TO MOVE IN ANY DIRECTION.

108. A free, material particle, acted upon by a system of forces which are not in equilibrium among themselves, will describe a path the direction of motion in which immediately after passing any point will depend upon its direction when it arrives at the point and the resultant of the forces acting upon it at the point, but the *direction of the acceleration* will be in the direction of the resultant of the impressed forces.

Let R be the resultant of the forces acting upon a particle whose mass is m at the point whose coördinates are x, y, z, and ds' the path which would be described by the effect of R only; then, according to Article 21, we have

$$R - m\frac{d^2s'}{dt^2} = 0.$$

Let a be the angle between the action-line of the resultant R (or of the arc ds') and the axis of x; multiplying by $\cos a$, we have

$$R\cos a - m\frac{d^2s'}{dt^2}\cos a = 0;$$

in which $R\cos a$ is the *x-component* of the resultant, and according to equation (51) equals X; or, in other words, it is the projection of the line representing the resultant on the axis of x; $d^2s'\cos a$ is the projection of d^2s' on the axis of x, and is d^2x. Hence, the equation becomes,

$$\left.\begin{aligned} &X - m\frac{d^2x}{dt^2} = 0; \\ \text{and similarly,}\quad &Y - m\frac{d^2y}{dt^2} = 0; \\ &Z - m\frac{d^2z}{dt^2} = 0; \end{aligned}\right\} \qquad (124)$$

which are the equations for the motion of a particle along the coördinate axes; and are also the equations for the motion of a body of finite size when the action-line of the resultant passes through the centre of the mass. They are also the *equations of translation* of the centre of any free mass when the forces produce both rotation and translation; in which case m should be changed to M to represent the total mass. See Article 38.

VELOCITY AND LIVING FORCE.

109. Multiplying the first of equations (124) by dx, the second by dy, and the third by dz, adding and reducing, give

$$Xdx + Ydy + Zdz = \tfrac{1}{2}md\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right) = \tfrac{1}{2}md\frac{ds^2}{dt^2};$$

and integrating gives

$$\int(Xdx + Ydy + Zdz) = \tfrac{1}{2}m\frac{ds^2}{dt^2} = \tfrac{1}{2}mv^2 + C.$$

The first member is the work done by the impressed forces; for if R be the resultant, and s the path, then, according to Article 25, equation (26), the work will be $\int Rds$, and by projecting this on the coördinate axes and taking their sum, we have the above expression. The second member is the stored energy plus a constant.

Let X, Y, Z be known functions of x, y, z, and that the terms are integrable. (It may be shown that they are always integrable when the forces act towards or from fixed centres.) Performing the integration between the limits x_0, y_0, z_0, and x_1, y_1, z_1, we have

$$\phi\,(x_0, y_0, z_0) - \phi\,(x_1, y_1, z_1) = \tfrac{1}{2}m\,(v^2{}_0 - v^2{}_1); \quad (125)$$

hence, the work done by the impressed forces upon a body in passing from one point to another equals the difference of the living forces at those points. It also appears that the velocity at two points will be independent of the path described; also, that, when the body arrives at the initial point, it will have the same velocity and the same energy that it previously had at that point.

EXAMPLES.

1. *If a body is projected into space, and acted upon only by gravity and the impulse; required the curve described by the projectile.*

Take the coördinate plane xy in the plane of the forces, x horizontal and y vertical, the origin being at the point from which the body is projected.

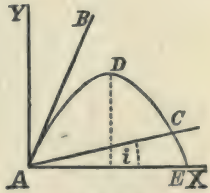


FIG. 109.

Let $W =$ the weight of the body;
$v =$ the velocity of projection; and
$a = BAX =$ the angle of elevation at which the projection is made.

We have,

$$X = 0; \qquad Y = -mg; \qquad Z = 0; \qquad z = 0;$$

and equations (124) become

$$\frac{d^2x}{dt^2} = 0;$$

$$-g - \frac{d^2y}{dt^2} = 0.$$

Integrating, observing that $v \cos a$ will be the initial velocity along the axis of x, and $v \sin a$ that along y, we have,

$$\frac{dx}{dt} = v \cos a;$$

$$\frac{dy}{dt} = v \sin a - gt;$$

and integrating again, observing that the initial spaces are zero we have,

$$\left.\begin{aligned} x &= vt \cos a; \\ y &= vt \sin a - \tfrac{1}{2}gt^2. \end{aligned}\right\} \qquad (a)$$

Eliminating t from these equations, gives

$$y = x \tan a - \frac{gx^2}{2v^2 \cos^2 a}; \qquad (b)$$

which is the equation of the common parabola, whose axis is parallel to the axis of y.

Let h be the height through which a body must fall to acquire a velocity v, then $v^2 = 2gh$, and the equation (b) becomes,

$$y = x \tan a - \frac{x^2}{4h \cos^2 a}. \qquad (c)$$

To find the range AE,
make $y = 0$ in equation (b), and we find $x = 0$, and

$$x = AE = 4h \cos a \sin a = 2h \sin 2a ; \qquad (d)$$

which is a maximum for $a = 45°$. The range will be the same for two angles of elevation, one of which is the complement of the other.

The greatest height,
will correspond to $x = h \sin 2a$, which, substituted in (b), gives,

$$h \sin^2 a. \qquad (e)$$

The velocity at the end of the time t
is.

$$V = \frac{ds}{dt} = \sqrt{\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right\}} = \sqrt{v^2 - 2v\,g\,t \sin a + g^2 t^2}; \qquad (f)$$

or by eliminating t by means of the first of equations (a), we have,

$$V = \sqrt{v^2 - 2g\,x \tan a + \frac{gx^2}{2h \cos^2 a}}. \qquad (g)$$

The direction of motion at any point
is found by differentiating equation (c), and making

$$\tan \theta = \frac{dy}{dx} = \tan a - \frac{x}{2h \cos^2 a}. \qquad (h)$$

At the highest point $\theta = 0$, $\therefore x = h \sin 2a$, as before found. For $x = 2h \sin 2a$, we have,

$$\tan \theta = - \tan a,$$

or the angle at the end of the range is the supplement of the angle of projection.

2. A body is projected at an angle of elevation of 45°, and has a range of 1,000 feet; required the velocity of projection, the time of flight, and the parameter of the parabola.

3. What must be the angle of elevation in order that the horizontal range may equal the greatest altitude? What, that it may equal n times the greatest altitude?

4. Find the velocity and the angle of elevation of a projectile, so that it may pass through the points whose coördinates are $x_1 = 400$ feet, $y_1 = 50$ feet, $x_2 = 600$ feet, and $y_2 = 40$ feet.

5. If the velocity is 500 feet per second, and the angle of elevation 45 degrees; required the range, the greatest elevation, the velocity at the highest point, the direction of motion 6,000 feet from the point of projection, and the velocity at that point.

6. If a plane, whose angle of elevation is i, passes through the origin, find the coördinates of the point C, Fig. 109, where the projectile passes it.

7. In the preceding problem, if i is an angle of depression, find the coördinates.

8. Find the equation of the path when the body is projected horizontally.

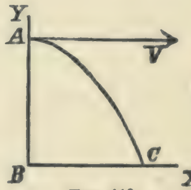


Fig. 110.

9. If a body is projected in a due southerly direction at an angle of elevation a, and is subjected to a constant, uniform, horizontal pressure in a due easterly direction; required the equations of the path, neglecting the resistance of the air.

We have

$$X = 0; \qquad Y = -mg; \qquad Z = F \text{ (a constant).}$$

The projection of the path on the plane xy will be a parabola, on xz also a parabola.

10. If a body is projected into the air, and the resistance of the air varies as the square of the velocity; required the equation of the curve.

(The final integrals for this problem cannot be found. Approximate solutions have been made for the purpose of determining certain laws in regard to gunnery. It is desirable for the student to establish the equations and make the first steps in the reduction.)

(The remainder of this chapter may be omitted without detriment to what follows it. It, however, contains an interesting topic in Mechanics, and is of vital importance in Mathematical Astronomy and Physics.)

CENTRAL FORCES.

110. Central forces are such as act directly towards or from a point called a centre. Those which act towards the centre are called *attractive*, and are considered negative, while those which act from the centre are *repulsive*, and are considered positive. The centre may be fixed or movable.

The line from the centre to the particle is called a *radius vector*. The path of a body under the action of central forces is called an orbit.

The forces considered in Astronomy and many of those in Physics, are central forces.

GENERAL EQUATIONS.

111. Consider the force as attractive, and let it be represented by $-F$.

Take the coördinate plane xy in the plane of the orbit, the origin being at the centre of the force, and $OP = r$, the radius vector, then

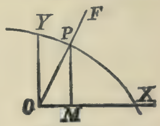


FIG. 111.

$$X = -F\cos\alpha = -F\frac{x}{r};$$

$$Y = -F\cos\beta = -F\frac{y}{r};$$

and the first two of equations (124) become

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= -F\frac{x}{r}; \\ m\frac{d^2y}{dt^2} &= -F\frac{y}{r}. \end{aligned}\right\} \tag{126}$$

To change these to polar coördinates, first modify them by multiplying the first by y and the second by x, and subtracting, and we have

$$my\frac{d^2x}{dt^2} - mx\frac{d^2y}{dt^2} = 0;$$

and multiplying the first by x and the second by y, and adding, we have

$$mx\frac{d^2x}{dt^2} + my\frac{d^2y}{dt^2} = -Fr.$$

Let $\theta = POM =$ the variable angle, then

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

and differentiating each twice, we find

$$d^2x = (d^2r - rd\theta^2)\cos\theta - (2drd\theta + rd^2\theta)\sin\theta;$$

$$d^2y = (d^2r - rd\theta^2)\sin\theta + (2drd\theta + rd^2\theta)\cos\theta;$$

which substituted in the preceding equations, give

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{F}{m}; \tag{127}$$

$$2\frac{drd\theta}{dt^2} + r\frac{d^2\theta}{dt^2} = 0,$$

which may be put under the form

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0. \tag{128}$$

Equation (127) shows that the acceleration along r is the force on a unit of mass; and (128) shows that there is no acceleration perpendicular to the radius vector.

112. *Principle of equal areas.*—Integrate equation (128), and we have

$$r^2\frac{d\theta}{dt} = C; \tag{129}$$

and integrating a second time, we have

$$\int r^2 d\theta = Ct; \tag{130}$$

the constant of integration being zero, since the initial values of t and θ are both zero. But from Calculus $\int r^2 d\theta$ is twice the sectoral area POX; hence the sectoral area swept over by the radius vector increases directly as the time; and *equal areas will be passed over in equal times.*

Making $t = 1$, we find that C will be twice the sectoral area passed over in a unit of time.

The converse is also true, *that if the areas are proportional to the times the force will be central.*

For, multiplying the first of (126) by y and the second by x and taking their difference, we have

$$m\frac{d^2x}{dt^2}y - m\frac{d^2y}{dt^2}x = 0;$$

or,

$$Xy - Yx = 0;$$

which is the equation of a straight line, and is the equation of the action-line of the resultant, and since it has no absolute term it passes through the origin.

113. *To find the equation of the orbit,* eliminate dt from equations (127) and (128). For the sake of simplifying the final equation, make $r = \frac{1}{u}$, and (129) becomes

$$\frac{1}{u^2} = C\frac{dt}{d\theta}. \tag{131}$$

Differentiating and reducing, gives

$$dr = -\frac{du}{u^2} = -C\frac{dudt}{d\theta};$$

or,

$$\frac{dr}{dt} = -C\frac{du}{d\theta}.$$

the first member of which is the velocity in the direction of the radius.

Differentiating again, gives

$$\frac{d^2r}{dt^2} = -C^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting these in equation (127) and at the same time making m equal to unity, since the unit is arbitrary, we have

$$-C^2u^2\frac{d^2u}{d\theta^2} - C^2u^3 = -F,$$

or,

$$\frac{d^2u}{d\theta^2} + u - \frac{F}{C^2u^2} = 0; \tag{132}$$

which is the differential equation of the orbit.

When the law of the force is known, the value of F may be substituted, and the equation integrated, and the orbit be definitely determined.

Multiplying by du and integrating the first two terms, we have

$$C^2\left(\frac{du^2}{d\theta^2} + u^2\right) - 2\int F\frac{du}{u^2} = C_1. \tag{133}$$

114. *Given the equation of the orbit, to find the law of the force.*

From equation (132), we have

$$F = C^2u^2\left(\frac{d^2u}{d\theta^2} + u\right). \tag{134}$$

Another expression is deduced as follows: let

p = the perpendicular from the centre on the tangent, then from Calculus we have

$$p^2 = \frac{r^4d\theta^2}{ds^2} = \frac{r^4d\theta^2}{dr^2 + r^2d\theta^2}$$

$$= \frac{1}{\frac{du^2}{d\theta^2} + u^2}. \tag{135}$$

Differentiating, gives

$$pdp = -\frac{\frac{d^2u}{d\theta^2} + u}{\left(\frac{du^2}{d\theta^2} + u^2\right)^2}du;$$

and dividing by p^4, substituting, $du = -u^2dr$, and reducing, we have

$$\frac{1}{p^3} \cdot \frac{dp}{dr} = u^2\left(\frac{d^2u}{d\theta^2} + u\right);$$

which, combined with equation (134), gives

$$F = \frac{C^2}{p^3} \cdot \frac{dp}{dr}; \qquad (136)$$

which is a more simple formula for determining the law of the central force.

115. *To determine the velocity at any point of the orbit.*
We have

$$v = \frac{ds}{dt} = \frac{ds}{dt}\frac{d\theta}{d\theta} = \frac{ds}{d\theta}\frac{d\theta}{dt}$$

$$= Cu^2 \frac{ds}{d\theta} \text{ (from equation (131))}$$

$$= \frac{C}{p}. \text{ (from } \textit{Dif. Cal.}\text{)}, \qquad (137)$$

Hence, *the velocity varies inversely as the perpendicular from the centre upon the tangent to the orbit.*

Another expression is found by substituting the value of p, equation (135), in (137).

Hence,

$$v = C\sqrt{\frac{du^2}{d\theta^2} + u^2}. \qquad (138)$$

Still another expression may be found by substituting equation (138) in (133); hence

$$v^2 = C_1 + 2\int F\frac{du}{u^2} \qquad (139)$$

$$= C_1 - 2\int F\,dr.$$

Since F is a function of r, the integral of this equation gives v in terms of r, or the velocity depends directly upon the

distance of the body from the centre. Hence, the velocity at any two points in the orbit is independent of the path between them, the *law* of the force remaining the some.

116. *To determine the time of describing any portion of the orbit.*

To find it in terms of r, eliminate $d\theta$ between equations (129) and (133), reduce and find

$$t = \int_{r_1}^{r_2} \frac{dr}{\sqrt{C_1 - \frac{C^2}{r^2} - 2\int F dr}}; \qquad (140)$$

which integrated gives the time.

To find it in terms of the angle, we have from (129)

$$t = \frac{1}{C}\int r^2 d\theta;$$

from which r must be eliminated by means of the equation of the orbit, and the integration performed in reference to θ.

117. *To find the components of the force along the tangent and normal.*

Let $T =$ the tangential component;
$N =$ the normal component;
and resolving them parallel to x and y, we have

$$m\frac{d^2x}{dt^2} = X = T\frac{dx}{ds} - N\frac{dy}{ds},$$

$$m\frac{d^2y}{dt^2} = Y = T\frac{dy}{ds} + N\frac{dx}{ds}.$$

Eliminating N gives

$$m\frac{d^2x}{dt^2}dx + m\frac{d^2y}{dt^2}dy = Tds;$$

or

$$T = X\frac{dx}{ds} + Y\frac{dy}{ds}, \text{ also } = m\frac{d^2s}{dt^2}.$$

Eliminating T gives

$$N = m\frac{dx}{ds}\frac{d^2y}{dt^2} - m\frac{dy}{ds}\frac{d^2x}{dt^2}$$

$$= \frac{mds^3}{dt^2ds}\left(\frac{dxd^2y - dyd^2x}{ds^3}\right)$$

$$= m\frac{v^2}{\rho}; \qquad (141)$$

hence *the component of the force in the direction of the normal is dependent entirely upon the velocity and radius of curvature.* This is called the *centrifugal force.* It is the measure of the force which deflects the body from the tangent. The force directed towards the centre is called *centripetal.*

If $\omega =$ the angular velocity described by the radius of curvature, then, $v = \rho\omega$, and equation (141) becomes

$$N = m\omega^2\rho. \qquad (142)$$

EXAMPLES.

1. If a body on a smooth, horizontal plane is fastened to a point in the plane by means of a string, what will be the number of revolutions per minute, that the tension of the string may be twice the weight of the body.

2. A body whose weight is ten pounds, revolves in a horizontal circle whose radius is five feet, with a velocity of forty feet per second; required the tension of the string which holds it. (Use equation (141).)

3. Required the velocity and periodic time of a body revolving in a circle at a distance of n radii from the earth's centre.

The weight of the body on the surface being mg, at the distance of n radii it will weigh $mg\left(\frac{r}{nr}\right)^2 = \frac{mg}{n^2}$, and this is a

measure of the force at that distance. (Use equation (141) or (139).)

$$Ans.\ v = \left(\frac{gr}{n}\right)^{\frac{1}{2}};\ t = 2\pi\left(\frac{n^3 r}{g}\right)^{\frac{1}{2}}.$$

(This is substantially the problem which Sir Isaac Newton used to prove the law of Universal Gravitation. See Whewell's *Inductive Sciences*.)

4. *A particle is projected from a given point in a given direction with a given velocity, and moves under the action of a force which varies inversely as the square of the distance from the centre; required the orbit.*

Let $\mu =$ the force at a unit's distance, then

$$F = \mu u^2,$$

and equation (132) becomes

$$\frac{d^2u}{d\theta^2} + u - \frac{\mu}{C^2} = 0;$$

or,

$$\frac{d^2}{d\theta^2}\left(u - \frac{\mu}{C^2}\right) = -\left(u - \frac{\mu}{C^2}\right);$$

the first integral of which becomes by reduction

$$d\theta = \frac{-d\left(u - \frac{\mu}{C^2}\right)}{\sqrt{A^2 - \left(u - \frac{\mu}{C^2}\right)^2}};$$

in which A is an arbitrary constant, and the negative value of the radical is used.

Integrating again, making θ_0 the arbitrary constant, we have

$$\theta - \theta_0 = \cos^{-1}\left\{\frac{u - \frac{\mu}{C^2}}{A}\right\};$$

which by reduction gives

$$u = \frac{1}{r} = \frac{\mu}{C^2}\left(1 + \frac{AC^2}{\mu}\cos(\theta - \theta_0)\right), \qquad (a)$$

which is the general polar equation of a conic section, the origin being at the focus. As this is the law of Universal Gravitation, it follows that the orbits of the planets and comets are conic sections having the centre of the sun for the focus. In equation (a), θ_0 is the angle between the major axis and a line drawn through the centre of the force, and $\frac{AC^2}{\mu}$ is the eccentricity $= e$; hence the equation may be written

$$u = \frac{\mu}{C^2}\Big(1 + e\cos(\theta - \theta_0)\Big). \qquad (b)$$

The magnitude and position of the orbit will be determined from the constants which enter the equation, and these are determined by knowing the position, velocity, and direction of motion at some point in the orbit.

Draw a figure to represent the orbit, and make a tangent to the curve at a point which we will consider the initial point. Let β be the angle between the path and the radius vector at the initial point, r_0 the initial radius vector, and V_0 the initial velocity; then at the initial point

$$u = \frac{1}{r_0}, \quad \theta = 0, \quad \cot\beta = \frac{dr}{r_0 d\theta} = -\frac{du}{ud\theta}, \qquad (c)$$

and from equation (b)

$$e\cos\theta_0 = \frac{C^2}{\mu r_0} - 1, \qquad (d)$$

$$\frac{du}{d\theta} = \frac{\mu e}{C^2}\sin\theta_0,$$

which, combined with equation (c), gives

$$\frac{C^2\cot\beta}{\mu r_0} = -e\sin\theta_0. \qquad (e)$$

From equation (137)

$$C = V_0 r_0 \sin\beta; \qquad (f)$$

which, substituted in equations (d) and (e), and the latter divided by the former, gives

$$\tan\theta_0 = \frac{V_0^2 r_0 \sin\beta\cos\beta}{\mu - V_0^2 r_0 \sin^2\beta}.$$

Squaring equations (d) and (e), adding and reducing by equation (f), give

$$e^2 = 1 - \frac{V_0^2 r_0^2 \sin^2\beta}{\mu}\left(\frac{2}{r_0} - \frac{V_0^2}{\mu}\right). \qquad (g)$$

Hence, when

$V_0^2 > \dfrac{2\mu}{r_0}$, $e > 1$, and the orbit is a hyperbola,

$V_0^2 = \dfrac{2\mu}{r_0}$, $e = 1$, and the orbit is a parabola,

$V_0^2 < \dfrac{2\mu}{r_0}$, $e < 1$, and the orbit is an ellipse.

or (see example 23, page 34) the orbit will be a hyperbola, a parabola, or an ellipse, according as the velocity of projection is greater than, equal to, or less than the velocity from infinity.

As the result of a large number of observations upon the planets, especially upon Mars, Kepler deduced the following laws:

1. The planets describe ellipses of which the Sun occupies a focus.

2. The radius vector of each planet passes over equal areas in equal times.

3. The squares of the periodic times of any two planets are as the cubes of the major axes of their orbits.

The first of these is proved by the preceding problem, since the orbits are reëntrant curves. The second is proved by equation (130). The third we will now prove.

5. *Required the relation between the time of a complete circuit of a particle in an ellipse, and the major axis of the orbit.*

Let the initial point be at the extremity of the major axis near the pole, then θ_0 in equation (d) will be zero, and we have

$$C^2 = \mu r_0 (1 + e);$$

but from the ellipse,

$$r_0 = a - ae = a(1 - e);$$

$$\therefore\ C = \sqrt{\mu a (1 - e^2)}. \qquad (a)$$

Equation (130) gives

$$T = \frac{2\ \textit{area of ellipse}}{C}$$

$$= \frac{2\pi a^2 \sqrt{(1 - e^2)}}{\sqrt{\mu a (1 - e^2)}}$$

$$= 2\pi \sqrt{\frac{a^3}{\mu}};$$

$$\therefore\ T^2 \propto a^3.$$

6. The orbit being an ellipse, required the law of the force The polar equation of the ellipse, the pole being at the focus, is

$$u = \frac{1}{r} = \frac{1 + e \cos(\theta - \theta_0)}{a(1 - e^2)};$$

which, differentiated twice, gives

$$\frac{d^2u}{d\theta^2} = -\frac{e \cos(\theta - \theta_0)}{a(1 - e^2)};$$

and these, in equation (132), give,

$$F = \frac{C^2}{a(1 - e^2)} \frac{1}{r^2};$$

hence *the force varies inversely as the square of the distance.*

7. Find the law of force by which the particle may describe a circle, the centre of the force being in the circumference of the circle. (Tait and Steele, *Dynamics of a Particle.*)

Ans. $F \propto \frac{1}{r^5}$.

8. If the force varies directly as the distance, and is attractive, determine the orbit.

(This is the law of molecular action, and analysis shows that the orbit is an ellipse. The problem is of great importance in Physics especially in Optics and Acoustics.)

CHAPTER IX.

CONSTRAINED MOTION OF A PARTICLE.

118. If a body is compelled to move along a given fixed curve or surface, it is said to be constrained. The given curve or surface will be subjected to a certain pressure which will be normal to it.

If instead of the curve or surface, a force be substituted for the pressure which will be continually normal to the surface, and whose intensity will be exactly equal and opposite to the pressure on the curve, the particle will describe the same path as that of the curve, and the problem may be treated as if the particle were free to move under the action of this system of forces.

Let $N =$ the normal pressure on the surface, and

$L = f(x, y, z) = 0$, be the equation of the surface;

$\theta_x, \theta_y, \theta_z$ the angles between N and the respective coördinate axes.

Then

$$\left.\begin{aligned} X + N\cos\theta_x - m\frac{d^2x}{dt^2} &= 0; \\ Y + N\cos\theta_y - m\frac{d^2y}{dt^2} &= 0; \\ Z + N\cos\theta_z - m\frac{d^2z}{dt^2} &= 0; \end{aligned}\right\} \qquad (143)$$

in which the third terms are the measures of the resultants of the axial components of the applied forces. We will confine the further discussion to forces in a plane. Take xy in the plane of the forces, then we have

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= X - N\frac{dy}{ds}; \\ m\frac{d^2y}{dt^2} &= Y + N\frac{dx}{ds}. \end{aligned}\right\} \qquad (144)$$

Eliminating N, we find

$$m\left\{\frac{dxd^2x}{dt^2}+\frac{dyd^2y}{dt^2}\right\}=Xdx+Ydy,$$

or

$$m\left\{d\left(\frac{dx}{dt}\right)^2+d\left(\frac{dy}{dt}\right)^2\right\}=2Xdx+2Ydy;$$

and integrating, making v_0 the initial velocity along the path, and v the velocity at any other point, we have

$$\tfrac{1}{2}m(v^2-v_0^2)=\int(Xdx+Ydy); \tag{145}$$

hence, *the living force gained or lost in passing from one point to another is equal to the work done by the impressed forces.* If the forces X and Y are functions of the coördinates x and y, and the terms within the parenthesis are integrable, the result may be expressed in terms of constants and functions of the coördinates of the initial and terminal points, and may be written $\quad \tfrac{1}{2}m(v^2-v_0^2)=c\phi(x_0,y_0)-c\phi(x,y)$;

hence, for such a system, *the velocity will be independent of the path described, and will be dependent only upon the coördinates of the points; also, the velocity will be independent of the normal pressure.*

119. *To find the normal pressure* multiply the first of equations (144) by dy, the second by dx, subtract, observing that $dx^2+dy^2=ds^2$, and we find

$$m\left\{\frac{dx}{ds}\frac{d^2y}{dt^2}-\frac{dy}{ds}\frac{d^2x}{dt^2}\right\}=Y\frac{dx}{ds}-X\frac{dy}{ds}+N;$$

or

$$m\frac{ds^2}{dt^2}\left\{\frac{dxd^2y-dyd^2x}{ds^3}\right\}=Y\frac{dx}{ds}-X\frac{dy}{ds}+N;$$

$$\therefore\ N=X\frac{dy}{ds}-Y\frac{dx}{ds}+m\frac{v^2}{\rho}; \tag{146}$$

in which ρ is the radius of curvature at the point. The first and second terms of the second member are the normal components of the impressed forces. *The total normal pressure*

will, therefore, be that due to the impressed forces plus that due to the force necessary to deflect the body from the tangent. The last term is called the *centrifugal force*, as stated in Article 117. If the body moves on the convex side of the curve, the last term should be subtracted from the others; hence it might be written $\pm m\frac{v^2}{\rho}$; in which $+$ belongs to movement on the concave arc, and $-$ on the convex.

120. *To find the time of movement,* from equation (145), we have

$$\frac{ds^2}{dt^2} - v_0^2 = \frac{2}{m}\int(Xdx + Ydy);$$

$$\therefore\ t = \int_{s_0}^{s}\frac{\sqrt{m}\,ds}{\sqrt{2\int(Xdx+Ydy)+mv_0^2}}. \qquad (147)$$

121. *To find where the particle will leave the constraining curve.*

At that point $N = 0$, which gives

$$m\frac{v^2}{\rho} = X\frac{dy}{ds} - Y\frac{dx}{ds}; \qquad (148)$$

which, combined with the equation of the curve, makes known the point.

If a body is subjected only to the force of gravity, we have $X = 0$ in all the preceding equations.

EXAMPLES.

1. *A body slides down a smooth inclined plane under the force of gravity; required the formulas for the motion.*

Take the origin at the upper end and let the equation of the plane be

$$y = ax;$$

y being positive downward. Then we have

$$X = 0, \qquad Y = mg, \qquad dy = a dx, \qquad v_0 = 0,$$

and equation (145) becomes

$$v^2 = 2 \int g a dx = 2gax = 2gy; \qquad (a)$$

hence, the velocity is the same as if it fell *vertically* through the same height.

To find the time, equation (147) gives

$$t = \int \frac{ds}{\sqrt{2gax}} = \sqrt{\frac{1 + a^2}{2ga}} \int \frac{dx}{\sqrt{x}} = s \sqrt{\frac{2}{g y}}; \qquad (b)$$

that is, *if the altitude of the plane* (y) *is constant the time varies directly as the length*, s.

We may also find

$$s = t \sqrt{\tfrac{1}{2} g y} = \tfrac{1}{2} g t^2 \sin a. \qquad (c)$$

2. Prove that the times of descent down all chords of a vertical circle which pass through either extremity of a vertical diameter are the same.

3. Find the straight line from a given point to a given inclined plane, down which a body will descend in the least time.

4. The time of descent down an inclined plane is twice that down its height; required the inclination of the plane to the horizon.

5. At the instant a body begins to descend an inclined plane, another body is projected up it with a velocity equal to the velocity which the first body will have when it reaches the foot of the plane; required the point where they will meet.

6. Two bodies slide down two inclined lines from two given points in the same vertical line to any point in a curve in the same time, the lines all being in one vertical plane; required the equation of the curve.

7. A given weight, P, draws another weight, W, up an inclined plane, by means of a cord parallel to the plane; through what distance must P act so that the weight, W, will move a feet after P is separated from it.

8. *Required a curve such that if it revolve with a uniform angular velocity about a vertical diameter, and a smooth ring of infinitesimal diameter be placed upon it at any point, it will not slide on the curve.*

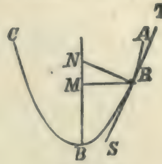


FIG. 112.

Let ω be the angular velocity, then we have

$$Y = -mg, \qquad X = m\omega^2 x, \qquad v = 0, \qquad v_0 = 0,$$

and equation (145) becomes

$$\omega^2 x^2 - 2gy + C = 0,$$

which is the equation of the common parabola.

If the origin be taken at B, C will be zero.

(ANOTHER SOLUTION.—Let NR be a normal to the curve, MR = the centrifugal force, NM = the force of gravity; but the latter is constant, hence NM, the subnormal, is constant, which is a property of the common parabola.)

9. Find the normal pressure against the curve in the preceding problem.

10. THE PENDULUM.—*Find the time of oscillation of the simple pendulum.* This is equivalent to finding the time of descent of a particle down a smooth arc of a vertical circle.

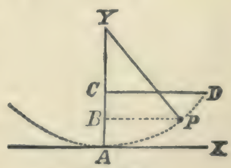


FIG. 113.

Take the origin of coördinates at A, the lowest point. Let the particle start at D, at a height $AC = h$; when it has arrived at P, it will have fallen through a height $CB = h - y$, and, according to equation (a) on the preceding page, will have a velocity

$$v = \sqrt{2g(h-y)} = \frac{ds}{dt} \qquad (a)$$

The equation of the arc is

$$x^2 = 2ry - y^2;$$

hence

$$dx^2 = \frac{(r-y)^2}{2ry - y^2} dy^2.$$

But

$$ds^2 = dx^2 + dy^2,$$

$$\therefore ds = \frac{r\,dy}{\sqrt{2ry - y^2}}.$$

Considering this as negative, since for the descent the arc is a decreasing function of the time, we have from (a)

$$t = \frac{r}{\sqrt{2g}} \int_0^h \frac{dy}{\sqrt{(h-y)(2ry-y^2)}}$$

This may be put in a form for integration by Elliptic Functions; but by developing it into a series, each term may be easily integrated. In this way we find

$$t = \tfrac{1}{2}\pi\sqrt{\frac{r}{g}}\left\{1 + \left(\tfrac{1}{2}\right)^2\frac{h}{2r} + \left(\tfrac{1}{2}\cdot\tfrac{3}{4}\right)^2\left(\frac{h}{2r}\right)^2 + \left(\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\right)^2\left(\frac{h}{2r}\right)^3 + \text{etc.}\right\};$$

by means of which the time may be approximated to, with any degree of accuracy.

When the arc is very small, all the terms containing $\frac{h}{2r}$ will be small, and by neglecting them, we have for a complete oscillation (letting l be the length of the pendulum),

$$T = 2t = \pi\sqrt{\frac{l}{g}}; \qquad (b)$$

that is, *for very small arcs the oscillations may be regarded as isochronal*, or performed in the same time.

For the same place *the times of vibration are directly as the square roots of the lengths of the pendulums.*

For any pendulum *the times of vibration vary inversely as the square roots of the force of gravity at different places.*

If t is constant

$$l \propto g.$$

11. What is the length of a pendulum which will vibrate three times in a second?

12. Prove that the lengths of pendulums vibrating during the

same time at the same place, are inversely as the squares of the number of vibrations.

13. Find the time of descent of a particle down the arc of a cycloid.

The differential equation of the curve referred to one end as an origin, x being horizontal and y vertical (r being the radius of the generating circle), is

$$dx = \frac{y}{\sqrt{2ry - y^2}}dy.$$

Ans. $\pi\sqrt{\frac{r}{g}}$.

The time will be the same from whatever point of the curve the motion begins, and hence, it is called *tautochronal.*

14. In the simple pendulum, find the point where the tension of the string equals the weight of the particle.

15. *A particle is placed in a smooth tube which revolves horizontally about an axis through one end of it; required the equation of the curve traced by the particle.*

The only force to impel the particle along the tube is the centrifugal force due to rotation.

Letting $r =$ the radius vector of the curve;
$r_0 =$ the initial radius vector;
$\omega =$ the uniform angular velocity;

we have

$$\frac{d^2r}{dt^2} = \omega^2 r;$$

which, integrated, gives

$$r = \tfrac{1}{2}r_0\,(e^{\omega t} + e^{-\omega t}),$$

hence, the relation between the radius vector and the arc described by the extremity of the initial radius vector, is the same as between the coördinates of a catenary. (See equation (k), p. 134.)

16. To find a curve joining two points down which a particle will slide by the force of gravity in the shortest time.

The curve is a cycloid. This problem is celebrated in the

history of Dynamics. The solution properly belongs to the Calculus of Variations, although solutions may be obtained by more elementary mathematics. Such curves are called *Brachistochrones.*

PROBLEMS PERTAINING TO THE EARTH.

122. *To find the value of g.*

We have, from example 10 of the preceding Article,

$$g = \frac{\pi^2 l}{T^2}. \tag{149}$$

Making $T = 1$ second and $l = 39.1390$ inches, the length of the pendulum vibrating seconds at the Tower of London, we have for that place,

$$g = 32.1908 \text{ feet.}$$

The relation between the force of gravity at different places on the surface of the earth is given in Article 19.

The determination of l depends upon the compound pendulum.

123. *To find the centrifugal force at the equator.*

We have, from equation (142), for a unit of mass,

$$f = \omega^2 R = \frac{4\pi^2}{T^2} R; \tag{a}$$

in which R, the equatorial radius, is 20,923,161 feet; T, the time of the revolution of the earth on its axis, is 86,164 seconds, and $\pi = 3.1415926$. These values give

$$f = 0.11126 \text{ feet.}$$

The force of gravity at the equator has been found to be 32.09022 feet (Article 19); hence, if it were not diminished by the centrifugal force, it would be

$$G = 32.09022 + 0.11126 = 32.20148 \text{ feet,}$$

and

$$\frac{f}{G} = \frac{0.11126}{32.20148} = \frac{1}{289} \text{ nearly;}$$

hence the centrifugal force at the equator is $\frac{1}{289}$ of the undiminished force of gravity.

Example.

In what time must the earth revolve that the centrifugal force at the equator may equal the force of gravity?

Ans. $\frac{1}{17}$ *of its present time.*

124. *To find the effect of the centrifugal force at different latitudes on the earth.*

Let $L = POQ =$ the latitude of the point P;

$R = OQ = OP =$ the radius of the earth;

then will the radius of the parallel of latitude PP' be

$$R_1 = R \cos L.$$

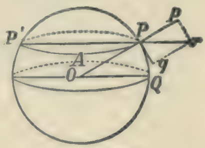


Fig. 114.

The centrifugal force will be in the plane of motion and may be represented by the line Pr, or

$$Pr = f_1 = \omega^2 R_1 = \omega^2 R \cos L;$$

therefore, the centrifugal force varies directly as the cosine of the latitude. But the force of gravity is in the direction PO. Resolving Pr parallel and perpendicular to PO, we have

$$Pp = \omega^2 R \cos^2 L = \tfrac{1}{289} G \cos^2 L;$$

$$Pq = \omega^2 R \cos L \sin L = \tfrac{1}{578} G \sin 2L;$$

the former of which diminishes directly the force of gravity, and the latter tends to move the matter in the parallel of latitude PP', toward the equator. Such a movement has taken place, and as a result the earth is an oblate spheroid. In the present form of the earth the action-line of the force of gravity is normal to the surface (or it would be if the earth were homogeneous), and hence, does not pass through the centre O, except on the equator and at the poles. The preceding formulas would be true for a rigid homogeneous sphere, but are only approximations in the case of the earth.

CHAPTER X.

FORCES IN A PLANE PRODUCING ROTATION.

125. ANGULAR MOTION OF A PARTICLE ABOUT A FIXED AXIS. Let the body C, on the horizontal arm AB, revolve about the vertical axis ED. Consider the body reduced to the centre of the mass, and the force F_1 applied at the centre and acting continually tangent to the path described by the particle. This may be done as shown in Fig. 121. In this case the force will be measured in the same way as if the path were rectilinear, for the force is applied along the path. Hence, according to equation (21),

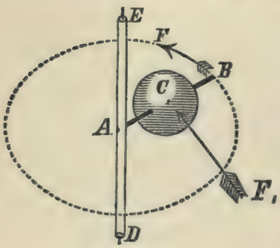


FIG. 115.

$$F_1 = m\frac{d^2s}{dt^2};$$

in which s is the arc of the circle.

If θ be the angle swept over by the radius, and r_1 the radius; then

$$s = r_1\theta,$$
$$ds = r_1\,d\theta,$$
$$d^2s = r_1\,d^2\theta;$$

which, substituted in the equation above, gives

$$F_1 = mr_1\frac{d^2\theta}{dt^2};$$

$$\therefore \frac{d^2\theta}{dt^2} = \frac{F_1}{mr_1}. \tag{150}$$

If a force, F, be applied to the arm AB, at a distance a from

the axis ED, producing the same movement of the mass C, we have, from the equality of moments, Article 65,

$$F_1 r_1 = Fa;$$

and the value of F_1 deduced from this equation substituted in the preceding one, gives

$$\frac{d^2\theta}{dt^2} = \frac{Fa}{mr_1^2}; \qquad (151)$$

that is, *the angular acceleration produced by a force, F, on a particle, m, equals the moment of the force divided by the moment of inertia of the mass.*

(For moments of inertia, see Chapter VII.)

We observe that, when the force is applied directly to the particle, it produces no strain upon the axis, but that it does when applied to other points of the arm. In both cases there will be a strain of

$$mr_1\left(\frac{d\theta}{dt}\right)^2 = mr_1\,\omega^2;$$

due to centrifugal force, ω being the angular velocity.

126. Angular motion of a finite mass. Let a body, AB, turn about a vertical axis at A, under the action of constant forces, F, acting horizontally; m_1, m_2, etc., masses of the elements of the body at the respective distances r_1, r_2, etc., from the axis A; and considering equation (151) as *typical*, we have

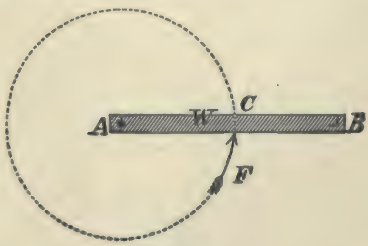


Fig. 116.

$$\frac{d^2\theta}{dt^2} = \frac{\Sigma Fa}{m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \text{etc.},} = \frac{\Sigma Fa}{\Sigma mr^2}$$

$$= \frac{\textit{moment of the forces}}{\textit{moment of inertia}}. \qquad (152)$$

127. Energy of a rotating mass. Multiply both members

of the preceding equation by $d\theta$, integrate and reduce, and we find

$$\tfrac{1}{2}\Sigma mr^2\left(\frac{d\theta}{dt}\right)^2 = \int F.rd\theta\,; \tag{153}$$

in which $rd\theta$ is an element of the space passed over by F, and $F.rd\theta$ is an element of work done by F; hence the second member represents the total work done by F upon the body, all of which is stored in it. Therefore, *the energy of a body rotating about an axis equals the moment of inertia of the mass multiplied by one-half the square of the angular velocity.*

If the body has a motion of translation and of rotation at the same time, the total energy will be the sum due to both motions; for it is evident that while a body is rotating a force may be applied to move it forward in space, Article 38, and that the work done by this force will be independent of the rotation.

If $v =$ the velocity of translation of the axis about which the body rotates;
$\omega =$ the angular velocity; and
$I_m =$ the moment of inertia of the *mass*;
then the total work stored in the body will be

$$\tfrac{1}{2}Mv^2 + \tfrac{1}{2}I_m\omega^2 \tag{154}$$

128. AN IMPULSE. Multiply both members of equation (152) by dt, integrate, and we find

$$\frac{d\theta}{dt} = \omega = \frac{\Sigma a\int Fdt}{\Sigma mr^2}.$$

But according to Article 27, $\int Fdt$ is the measure of an impulse and is represented by Q, hence

$$\omega = \frac{Qa}{\Sigma mr^2} \tag{155}$$

$$= \frac{\textit{moment of the impulse}}{\textit{moment of inertia}}$$

129. THE TIME required to pass over n circumferences will be, in the case of an impulse,

$$t = \frac{2n\pi}{\omega} = \frac{2n\pi I}{Qa}.$$

In the case of accelerated forces the time will be found by integrating equation (153).

130. Simultaneous movement of rotation and of translation. If the body be unconstrained, the motion of translation of the body will be that of the centre of the mass. If the axis of rotation is rigid, it may be located anywhere in the body, or even without the body by considering it as rigidly connected with the body, in which case the motion of translation will be that of some point of the axis. In either case the motion of translation may be considered as resulting from a force acting directly upon the axis of rotation, and the rotation, by a force acting at some other point. The two motions may then be considered as existing independently of each other.

131. Formulas for the movement of a body involving both translation and rotation. The *general* equations for this case are (164) and (165) in the next chapter. In this Article let the rotation be about an axis parallel to z, and the centre of the mass move in the plane xy. Resolve the forces into couples and forces applied at the origin of coördinates, as in Article 83; then will the third of equations (86) be the impressed forces which produce rotation. Let R be the resultant of the forces at the origin at any instant, (see Article 84), and s be measured along the path described by the point of intersection of the axis of rotation with the plane xy. Then will the principles of Article 21, give

$$\left.\begin{aligned} R - \Sigma m \frac{d^2 s}{dt^2} &= 0; \\ \Sigma (Xy - Yx) - \Sigma \left(my \frac{d^2 x}{dt^2} - mx \frac{d^2 y}{dt^2} \right) &= 0; \end{aligned}\right\} \qquad (156)$$

The expression $\Sigma (Xy - Yx)$ is the sum of the moments of the impressed forces $= \Sigma Fa$ (Article 60). Transforming the second term of the last equation into polar coördinates having the same origin, we have

$$\Sigma \left(my \frac{d^2 x}{dt^2} - mx \frac{d^2 y}{dt^2} \right) = \Sigma mr^2 \frac{d^2 \theta}{dt^2};$$

hence, the equations become

$$\left.\begin{aligned} m\frac{d^2s}{dt^2} &= F; \\ \frac{d^2\theta}{dt^2} &= \frac{\Sigma Fa}{\Sigma mr^2}. \end{aligned}\right\} \qquad (157)$$

132. Reduced mass. A given mass may be concentrated at such a point, or in a thin annulus, that the force or impulse will have the same effect upon it as if it were distributed. To accomplish this it is only necessary that Σmr^2 in the second of (157) should have an equivalent value. Let M be the mass of the body, k the distance from the axis to the required point, then

$$\Sigma mr^2 = Mk^2,$$

in which k is the radius of gyration, as defined in Article 107.

But any other point may be assumed, and a *mass determined* such that the effect shall be the same. Let κ be the distance to that point (or radius of the annulus), and M_1 the required mass then we have

$$Mk^2 = M_1\kappa^2;$$

$$\therefore\ M_1 = M\frac{k^2}{\kappa^2}; \qquad (158)$$

which is called the reduced mass.

EXAMPLES.

1. *A slender bar AB, falls through a height h, retaining its horizontal position until one end strikes a fixed obstacle C; required the angular velocity of the piece and the linear velocity of the centre immediately after the impulse.*

Let M be the mass of the bar, l its length, v the velocity of the centre at the instant of impact, and v_1 the velocity of the centre immediately after impact. Consider the bodies as perfectly non-elastic; then will the effect of the impact be simply to suddenly arrest the end A.

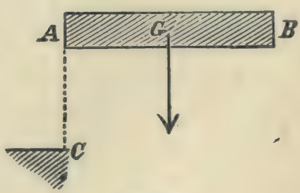


Fig. 117.

The bar will rotate about a horizontal axis through the centre, as shown by Article 38; and, as shown by Articles 27 and 38, the

impulse will be $Q = M(v - v_1)$; that is, *it is the change of velocity at the centre multiplied by the mass.* The impact will entirely arrest the motion of the end, A, at the *instant* of the impact, and hence at that instant the angular velocity of A in reference to G will be the same as G in reference to A.

Equation (155) gives

$$\omega = \frac{\textit{moment of impulse}}{\textit{moment of inertia}}$$

$$= \frac{M(v - v_1)\frac{1}{2}l}{\frac{1}{12}Ml^2}$$

$$= 6\frac{v - v_1}{l}.$$

But at the instant of the impact

$$v_1 = \tfrac{1}{2}l\omega,$$

solving these give

$$\omega = \tfrac{3}{2}\frac{v}{l}, \quad v_1 = \tfrac{3}{4}v.$$

We now readily find

$$Q = \tfrac{1}{4}Mv.$$

To find the velocity of any point in a vertical direction at the instant of the impact, we observe that it may be considered as composed of two parts; a linear velocity v_1 downward, and a right-handed rotation. The actual velocity at A due to rotation will be

$$\tfrac{1}{2}l\omega = \tfrac{3}{4}v,$$

which will be upward, and the linear velocity downward will be $v_1 = \frac{3}{4}v$, hence the result will be no velocity. Similarly, the velocity at B will be $\frac{3}{4}v + \frac{3}{4}v = \frac{3}{2}v$. Also, for any point distant x from G, we have at the left of G

$$\tfrac{3}{4}v - \omega x = \tfrac{3}{4}v\left(1 - 2\frac{x}{l}\right);$$

and to the right of G we have

$$\tfrac{3}{4}v\left(1 + 2\frac{x}{l}\right).$$

When the bar comes into a vertical position, we easily find

that A has passed below a horizontal through C. Every point, therefore, has a progressive velocity, except the point A, at the instant of impact.

After the impact the centre will move in the same vertical and with an accelerated velocity, while the angular velocity will remain constant.

2. Suppose that impact takes place at one-quarter the length from A, required the angular velocity.

3. At what point must the impulse be made so that the velocity of the extremity B will be doubled at the instant of impact?

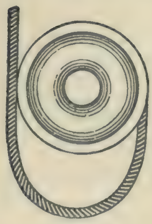


FIG. 118.

4. An inextensible string is wound around a cylinder, and has its free end attached to a fixed point. The cylinder falls through a certain height (not exceeding the length of the free part of the string), and at the instant of the impact the cord is vertical and tangent to the cylinder; all the forces being in a plane; required the angular velocity produced by the impulse, and the momentum.

$$Ans.\ \tfrac{2}{3}\frac{v}{r}; \quad Q = \tfrac{1}{3}Mv.$$

5. In the preceding problem, let the body be a homogeneous sphere, the string being wound around the arc of a great circle.

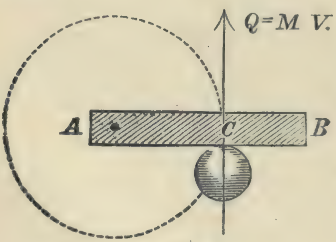


FIG. 119.

6. *A homogeneous prismatic bar AB, in a horizontal position constrained to revolve about a vertical fixed axis A, receives a direct impulse from a sphere whose momentum is Mv; required the angular velocity of the bar.*

The momentum imparted to the bar will depend upon the elasticities of the two bodies. Consider them perfectly elastic. The effect of the impulse will be the same as if the mass of the bar were concentrated at the extremity of the radius of gyration; hence an equivalent mass at the point C may be determined.

Let M_1 = the mass of AB;

M_2 = the reduced mass;

v_2 = the velocity of the reduced mass after impact;

$a = AC$.

Then, by equation (158), the mass of the bar reduced to the point C, will be

$$M_2 = M_1 \frac{k^2}{a^2}.$$

By equation (40) the velocity of M_2 after impact will be

$$v_2 = \frac{2M}{M + M_2} v = \frac{2Ma^2}{Ma^2 + M_1 k^2} v; \tag{a}$$

hence, the momentum imparted will be

$$M_2 v_2 = \frac{2MM_1 k^2}{Ma^2 + M_1 k^2} v;$$

and the moment will be

$$M_2 v_2 a = \frac{2MM_1 k^2 a}{Ma^2 + M_1 k^2} v.$$

According to equation (155), we have

$$\omega = \frac{\textit{moment of the impulse}}{\textit{moment of inertia}}$$

$$= \frac{\dfrac{2MM_1 k^2 a}{Ma^2 + M_1 k^2} v}{M_1 k^2}$$

$$= \frac{2Ma}{Ma^2 + M_1 k^2} v.$$

This result is the same as that found by dividing equation (a) by a, as it should be.

7. *Suppose, in the preceding problem, that there is no fixed axis, but that the body is free to translate; find where the impact must be made that the initial velocity at the end A shall be zero.*

Let Mv be the impulse *imparted* to the body;

$Mk_1^2 =$ the principal moment of inertia;

$h =$ the distance from the centre of the bar to the required point;

then

$$\omega = \frac{\textit{moment of the impulse}}{\textit{moment of inertia}}$$

$$= \frac{Mvh}{Mk_1^2} = \frac{vh}{k_1^2}; \qquad (a)$$

and the movement at A in the circular arc will be

$$\tfrac{1}{2}l\omega = \frac{vlh}{2k_1^2};$$

and the initial linear movement will be

$$v - \frac{vlh}{2k_1^2};$$

which, by the conditions of the problem, will be zero; hence

$$\frac{vlh}{2k_1^2} = v$$

or,

$$h = \frac{2k_1^2}{l}. \qquad (b)$$

The distance from A will be

$$h + \tfrac{1}{2}l = \frac{k_1^2 + (\frac{1}{2}l)^2}{\frac{1}{2}l}.$$

If the bar be of infinitesimal section

$$k_1^2 = \tfrac{1}{12}l^2;$$

$$\therefore h + \tfrac{1}{2}l = \tfrac{2}{3}l.$$

The result is independent of the magnitude of the impulse. From (b) we have

$$h(\tfrac{1}{2}l) = k_1^2;$$

hence, h and $\frac{1}{2}l$ are convertible, and we infer that if the impulse be applied at A the point of no initial motion will be at

the point given by equation (b), where the impact was previously applied.

8. *In the preceding problem find where the impulse must be applied so that the point of no initial velocity shall be at a distance h' from the centre.*

The initial linear velocity due to the rotary movement found from (a) of the preceding example, will be

$$h'\omega = \frac{v}{k_1^2} hh',$$

and the initial movement of the required point being zero, we have

$$v - \frac{v}{k_1^2} hh' = 0;$$

$$\therefore\ h = \frac{k_1^2}{h'}. \tag{159}$$

If the point of impact be at b, the point a, where the initial movement is zero, will be on the other side of the centre of the body. Let B be the centre, then

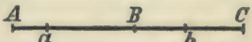


FIG. 120.

$$h = bB, \qquad h' = aB,$$

and from (159), we have

$$hh' = bB.aB = k_1^2; \tag{160}$$

and as k_1 is a constant, *the points a and b are convertible.*

AXIS OF SPONTANEOUS ROTATION.

133. In the preceding problem the *initial* motion would have been precisely the same if there had been a fixed axis through a perpendicular to the plane of motion, and hence the *initial motion* may be considered as a rotation about that axis. If a fixed axis were there it evidently would not receive any shock from the impulse.

The axis about which a quiescent body tends to turn at the instant that it receives an impulse is called the axis of spontaneous rotation.

CENTRE OF PERCUSSION.

134. When there is a fixed axis and the body is so struck that there is no impulse on the axis, *any point in the action-line of the force is called the centre of percussion.* Thus in Fig. 120, if a is the fixed axis, b will be the centre of percussion. It is also evident that, if b be a fixed object, and it be struck by the body AC, rotating about a, the axis will not receive an impulse.

AXIS OF INSTANTANEOUS ROTATION.

135. An axis through the centre of the mass, parallel to the axis of spontaneous rotation, is called the *axis of instantaneous rotation.* A free body rotates about this axis.

In regard to the spontaneous axis, we consider *that* as fixed in space for the instant; but at the same time the body really rotates about the instantaneous axis which moves in space with the body.

EXAMPLES UNDER THE PRECEDING EQUATIONS CONTINUED.

9. *A horizontal uniform disc is free to revolve about a vertical axis through its centre. A man walks around on the outer edge; required the angular distance passed over by the man and disc when he has walked once around the circumference.*

Let $W =$ the weight of the man;

$w =$ the weight of the disc;

$r =$ the radius of the disc;

$\omega_1 =$ the angular velocity of the man in reference to a fixed line;

$\omega =$ the angular velocity of the disc in reference to the same fixed line;

$Q =$ the force exerted by the man against the disc;

$k_1^2 = \frac{1}{2}r^2$.

The result will be the same whether the effort be exerted suddenly, or with a uniform acceleration, or irregularly. We will, therefore, treat it as if it were an impulse. The weights are here used instead of the masses, for they are directly pro-

portional to each other, and it is more natural to speak of the weight of a man than the mass of a man.

We have

$$\omega = \frac{\textit{moment of the impulse}}{\textit{moment of inertia}}$$

$$= \frac{Qr}{wk_1^2}g$$

$$= \frac{Wvr}{wk_1^2}$$

$$= \frac{2W}{w}\omega_1.$$

If, in a unit of time the man arrives at the initial point of the disc, we have

$$\omega + \omega_1 = 2\pi;$$

which, combined with the preceding equation, gives

$$\omega_1 = \frac{2w\pi}{w + 2W}.$$

If $W = w$, we have

$$\omega_1 = \tfrac{2}{3}\pi,$$

for the angular space passed over by the man, and

$$\omega = \tfrac{4}{3}\pi,$$

for the distance passed over by the disc.

10. *In Fig. 115 let the force F be constant; required the number of complete turns which the body C will make about the axis DE in the time t.*

Let r = the radius of the circle passed over by F;

r_1 = the distance of the centre of the body from the axis of revolution;

k_1 = the principal radius of gyration of the body in reference to a moment axis parallel to DE;

k = the radius of gyration of the body in reference to the axis DE;

then, according to equation (123),

$$k^2 = r_1^2 + k_1^2;$$

and, according to equation (152),

$$\frac{d^2\theta}{dt^2} = \frac{moment\ of\ forces}{moment\ of\ inertia}$$

$$= \frac{Fr}{Mk^2}.$$

Multiply by dt and integrate, and we have

$$\frac{d\theta}{dt} = \frac{Fr}{Mk^2}t.$$

the constant being zero, for the initial quantities are zero.

Multiplying again by dt, we find

$$\theta = \frac{Fr}{2Mk^2}t^2;$$

which is the angular space passed over in time t; and the number of complete rotations will be

$$\frac{\theta}{2\pi} = \frac{Frt^2}{4\pi Mk^2}.$$

11. If the body were a sphere 2 feet in diameter, weighing 100 pounds, the centre of which was 5 feet from the axis; F, a force of 25 pounds, acting at the end of a lever 8 feet long; required the number of turns which it will make about the axis in 5 minutes.

12. If the data be the same as in the preceding example; required the time necessary to make one complete turn about the axis.

13. Suppose that an indefinitely thin body, whose weight is W, rests upon the rim of a horizontal pulley which is perfectly free to move. A string is wound around the pulley, and passes over another pulley and has a weight, P, attached to its lower end. Supposing that there is no resistance by the pulleys or the string, required the distance passed over by P in time t.

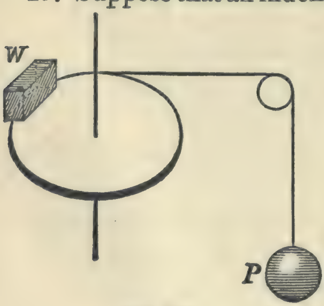


FIG. 121

According to equation (152), we have

$$\frac{d^2\theta}{dt^2} = \frac{Pr}{\frac{W + P}{g} r^2} = \frac{Pg}{(W + P) r};$$

from which it may be solved.

(This is equivalent to applying the weight P directly to the weight W, as in Fig. 10, and hence we have, according to equation (21),

$$\frac{W + P}{g} \frac{d^2s}{dt^2} = P;$$

but referring it to polar coördinates, we have $r \frac{d^2\theta}{dt^2} = \frac{d^2s}{dt^2}$, which substituted reduces the equation directly to that in the text.)

14. A disc whose weight is W is free to revolve about a horizontal axis passing through its centre and perpendicular to its plane. A cord is wound around its circumference and has a weight, P, attached to its lower end; required the distance through which P will descend in t seconds.

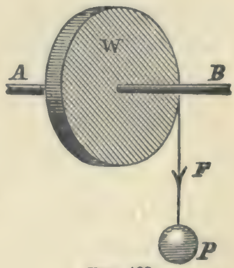


FIG. 122

We have

$$\frac{d^2\theta}{dt^2} = \frac{Prg}{Wk_1^2 + Pr^2};$$

from which θ may be found, and the space will be $r\theta$.

(This may be solved by equation (21). The mass of the disc reduced to an equivalent at the circumference will be $\frac{W}{g} \frac{k_1^2}{r^2}$, and that equation will become $\frac{1}{g}\left(P + W\frac{k_1^2}{r^2}\right)\frac{d^2s}{dt^2} = P$; which, by changing to polar coördinates, may be reduced to the equation in the text.)

15. If, in the preceding example, the body were a sphere revolving about a horizontal axis, the diameter of the sphere being 16 inches, weight 500 pounds, moved by a weight of 100 pounds descending vertically, the cord passing around a groove in the sphere the diameter of which is one foot; required the number of revolutions of the sphere in five seconds.

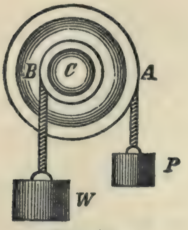


Fig. 123.

16. Two weights, P and W, are suspended on two pulleys by means of cords, as shown in Fig. 123, the pulleys being attached to the same axis C. No resistance being allowed for the pulleys, axle, or cords; required the circumstances of motion.

We have

$$\frac{d^2\theta}{dt^2} = \frac{\textit{moment of the forces}}{\textit{moments of inertia}}$$

$$= \frac{P.AC - W.BC}{P(AC)^2 + W(BC)^2 + disc\,AC.k_1^2 + disc\,BC.k_2^2} g;$$

in which *disc* AC, etc., are used for the weights of the discs. Let the right-hand member be represented by M, then we have

$$\frac{d\theta}{dt} = \omega = Mt;$$

$$\theta = \tfrac{1}{2}Mt^2.$$

17. In the preceding example let the discs be of uniform density, $AC = 8$ inches, $BC = 3$ inches; the weight of $AC = 6$ pounds, of $CB = 2$ pounds, of $P = 25$ pounds, and of $W = 60$ pounds; if they start from rest, required the space passed over by P in 10 seconds, and the tension of the cords.

18. *A homogeneous, hollow cylinder rolls down an inclined plane by the force of gravity; required the time.*

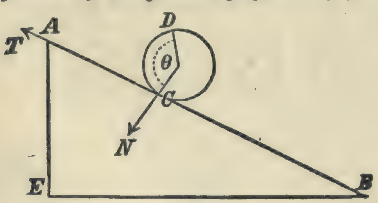


Fig. 124.

The weight of the cylinder may be resolved into two components, one parallel to the plane, which impels the body down it, the other normal, which induces friction. The friction acts parallel to the plane and tends to prevent the movement down it, and is assumed to be sufficient to prevent sliding.

Let $W =$ the weight of the cylinder;

$i =$ the inclination of the plane to the horizontal;

$N = W \cos i =$ the normal component;

$m =$ the mass of a unit; the altitude $= 1$;

$\phi =$ the coefficient of friction;
$T = \phi N =$ the tangential component;
$T' = W \sin i =$ the component of the weight parallel to the plane;
$r_1 =$ the internal radius of the cylinder;
$r =$ the external radius;
$\theta =$ the angular space passed over by the radius;
$s = AC$, the space.

This is a case of translation and rotation combined, and equations (157) give

$$\frac{W}{g}\frac{d^2s}{dt^2} = T' - T = W \sin i - T;$$

$$\frac{d^2\theta}{dt^2} = \frac{T.r}{\frac{1}{2}m\pi\,(r^4 - r_1^4)} = \frac{2gTr}{W\,(r^2 + r_1^2)};$$

and from the problem

$$s = r\theta.$$

Eliminating s and T from these equations, we get

$$\frac{d^2\theta}{dt^2} = \frac{2gr \sin i}{3r^2 + r_1^2}.$$

Integrating and making the initial spaces zero, we have

$$\theta = \frac{gr \sin i}{3r^2 + r_1^2}\,t^2;$$

$$\therefore\ t = \sqrt{\frac{3r^2 + r_1^2}{gr^2 \sin i}\,s}.$$

If $r_1 = 0$, the cylinder will be solid, and

$$t = \sqrt{\frac{3s}{g \sin i}},$$

and hence, the time is independent of the diameter of the cylinder.

If $r_1 = r$, the cylinder will be a thin annulus, and

$$t = \sqrt{\frac{4s}{g \sin i}};$$

hence, the time of descent will be $\sqrt{\frac{4}{3}}$ times as long as

when the cylinder is solid; the weight being the same in both cases.

If it slide down a smooth plane of the same slope, we have

$$t = \sqrt{\frac{2s}{g \sin i}},$$

which is less than either of the two preceding times.

THE PENDULUM.

FIG. 125

19. *Let a body be suspended on a horizontal axis and moved by the force of gravity; required the circumstances of motion.*

We have

$$-\frac{d^2\theta}{dt^2} = \frac{\textit{moment of forces}}{\textit{moment of inertia}}$$

$$= \frac{Wh \sin \theta}{Mk^2};$$

in which

$h = Oa$, the distance from the axis of suspension to the centre of gravity a of the body;

$W =$ the weight of the body;

$\theta = bOa$; and let

$k_1 =$ the principal radius of gyration;

then the preceding equation becomes

$$-\frac{d^2\theta}{dt^2} = \frac{gh}{h^2 + k_1^2} \sin \theta.$$

This equation cannot be completely integrated in finite terms, but by developing $\sin \theta$ and neglecting all powers above the first, we find for a complete oscillation

$$T = \pi \sqrt{\frac{h^2 + k_1^2}{gh}}; \qquad (161)$$

which gives the time in seconds when h, k_1 and g are given in feet.

To find the length of a simple pendulum which will vibrate

in the same time, we make equations (*b*), page 196, and (161) equal to one another, and have

$$l = \frac{h^2 + k_1^2}{h} = Od. \qquad (162)$$

Let $ad = h_1$, then

$$l - h = h_1 = \frac{k_1^2}{h};$$

$$\therefore\ hh_1 = k_1^2 \qquad (163)$$

136. Definitions. A body of any form oscillating about a fixed axis is called a *compound pendulum.*

A material particle suspended by a string without weight, oscillating about a fixed axis, is called a *simple pendulum.*

The point d is called the *centre of oscillation.* It is the point at which, if a particle be placed and suspended from the axis O by a string without weight, it will oscillate in the same time as the body Od. Or, it is the point at which, if the entire mass be concentrated, it will oscillate about the axis in the same time as when it is distributed.

The point O, where the axis pierces the plane xy, is called the *centre of suspension.*

137. Results. The centres of oscillation and of percussion coincide. (See Article 134.)

According to equation (163), the centres of oscillation and of suspension are convertible.

According to the same equation the principal radius of gyration is a mean proportional between the distances of the centres of oscillation and of suspension from the centre of gravity.

Equation (161) indicates a practical mode of determining the principal radius of gyration. To find it, let the body oscillate, and thus find T, then attach a pair of spring balances to the lower end and bring the body to a horizontal position, and find how much the scales indicate; knowing which, the weight of the body and the distance between the point of attachment and the centre of suspension O, the value of h may easily be computed. The value of g being known, all the quantities in equation (161) become known except k_1, which is readily found by a solution of the equation.

Examples.

1. A prismatic bar oscillates about an axis passing through one end, and perpendicular to its length; required the length of an equivalent simple pendulum.

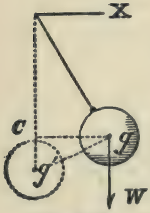


Fig. 126.

2. A homogeneous sphere is suspended from a point by means of a fine thread, find the length of a simple pendulum which will oscillate in the same time.

138. Captain Kater used the principle of the convertibility of the centres of suspension and oscillation for determining the length of a simple seconds pendulum, and hence the acceleration due to gravity.—*Phil. Trans.*, 1818.

Let a body, furnished with a movable weight, be provided with a point of suspensiou C (figure not shown), and another point on which it may vibrate, fixed as nearly as can be estimated in the centre of oscillation O, and in a line with the point of suspension and the centre of gravity. The oscillations of the body must now be observed when suspended from C and also when suspended from O. If the vibrations in each position should not be equal in equal times, they may readily be made so by shifting the movable weight. When this is done, the distance between the two points C and O is the length of the simple equivalent pendulum. Thus the length L and the corresponding time T of vibration will be found uninfluenced by any irregularity of density or figure. In these experiments, after numerous trials of spheres, etc., knife edges were preferred as a means of support. At the centres of suspension and oscillation there were two triangular apertures to admit the knife edges on which the body rested while making its oscillations.

Having thus the means of measuring the length L with accuracy, it remains to determine the time T. This is effected by comparing the vibrations of the body with those of a clock. The time of a single vibration or of any small arbitrary number of vibrations cannot be observed directly, because this would require the fraction of a second of time, as shown by the clock, to be estimated either by the eye or ear. The vibrations of the

body may be counted, and here there is no fraction to be estimated, but these vibrations will not probably fit in with the oscillations of the clock pendulum, and the differences must be estimated. This defect is overcome by "the method of coincidences." Supposing the time of vibration of the clock to be a little less than that of the body, the pendulum of the clock will gain on the body, and at length at a certain vibration the two will for an instant coincide. The two pendulums will now be seen to separate, and after a time will again approach each other, when the same phenomenon will take place. If the two pendulums continue to vibrate with perfect uniformity, the number of oscillations of the pendulum of the clock in this interval will be an integer, and the number of oscillations of the body in the same interval will be less by one complete oscillation than that of the pendulum of the clock. Hence by a simple proportion the time of a complete oscillation may be found.

The coincidences were determined in the following manner: Certain marks made on the two pendulums were observed by a telescope at the lowest point of their arcs of vibration. The field of view was limited by a diaphragm to a narrow aperture across which the marks were seen to pass. At each succeeding vibration the clock pendulum follows the other more closely, and at last the clock-mark completely covers the other during their passage across the field of view of the telescope. After a few vibrations it appears again preceding the other. The time of disappearance was generally considered as the time of coincidence of the vibrations, though in strictness the mean of the times of disappearance and reappearance ought to have been taken, but the error thus produced is very small. (*Encyc. Met.* Figure of the Earth.) In the experiments made in Hartan coal-pit in 1854, the Astronomer Royal used Kater's method of observing the pendulum. (*Phil. Trans.*, 1856.)

The value of T thus found will require several corrections. These are called "Reductions." If the centre of oscillation does not describe a cycloid, allowance must be made for the alteration of time depending on the arc described. This is called "the reduction to infinitely small arcs." If the point of support be not absolutely fixed, another correction is required

(*Phil. Trans.*, 1831). The effect of the buoyancy and the resistance of the air must also be allowed for. This is the "reduction to a vacuum." The length L must also be corrected for changes of temperature.

The time of an oscillation thus corrected enables us to find the value of gravity at the place of observation. A correction is now required to reduce this result to what it would have been at the level of the sea. The attraction of the intervening land must be allowed for by Dr. Young's rule (*Phil. Trans.*, 1819). We thus obtain the force of gravity at the level of the sea, supposing all the land above this level were cut off and the sea constrained to keep its present level. As the sea would tend in such a case to change its level, further corrections are still necessary if we wish to reduce the result to the surface of that spheroid which most nearly represents the earth. (See *Camb. Phil. Trans.*, vol. x.)

There is another use to which the experimental determination of the length of a simple equivalent pendulum may be applied. It has been adopted as a standard of length on account of being invariable and capable at any time of recovery. An Act of Parliament, 5 Geo. IV., defines the yard to contain thirty-six such parts, of which parts there are 39.1393 in the length of the pendulum vibrating seconds of mean time in the latitude of London, in vacuo, at the level of the sea, at temperature 62° F. The Commissioners, however, appointed to consider the mode of restoring the standards of weight and measure which were lost by fire in 1834, report that several elements of reduction of pendulum experiments are yet doubtful or erroneous, so that the results of a convertible pendulum are not so trustworthy as to serve for supplying a standard for length; and they recommend a material standard, the distance, namely, between two marks on a certain bar of metal under given circumstances, in preference to any standard derived from measuring phenomena in nature. (*Report*, 1841.)

All nations, practically, use this simple mode of determining the length of the standard of measure, that of placing two marks on a bar, and by a legal enactment declaring it to be a certain length.

139. Form of the Earth. The pendulum furnishes one of the best means for determining the form of the earth.

Let a = the equatorial radius of the earth;
b = the semi-axis;
ϵ = the ellipticity of the earth;

then

$$\epsilon = \frac{a-b}{a}.$$

Let m = ratio of the centrifugal force at the equator to the force of gravity at the same place;
l_0 = length of a second's pendulum at the equator;
l_{90} = the length of a second's pendulum at the poles;

then, from the *Mécanique Céleste*, tome II., No. 34, we have

$$\epsilon = \tfrac{5}{2}\, m - \frac{l_{90} - l_0}{l_0}.$$

The value of m is $\frac{1}{289}$. The formula for the length of the second's pendulum when the length at Paris is taken as unity, is

$$l = 0.996823 + 0.00549745 \sin^2 H,$$

when H is the latitude of the place. See Puissant's *Traité de Géodésie*, page 461.

By this means it has been found that ϵ is about $\frac{1}{311}$. Bowdich, in his translation of the *Mécanique Céleste*, p. 485, remarks, "It appears that the oblateness (ϵ) does not differ much from $\frac{1}{300}$, and may possibly be a little more, though some results give a little less."

140. Torsion Pendulum. If an elastic bar, CD, be fixed at one end, and at the other end have two weights, A_1 and A_2, rigidly fixed to it by means of the cross arm, $A_1 A_2$, then if the arm be turned into the position B_1B_2, the elastic resistance of the bar DC will cause the weights to move back to A_1A_2, and by virtue of the energy of the weights at that point, they will pass that position, and move on until their motion is arrested by the action of the elastic resistance of the

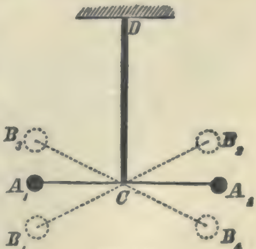


Fig. 127.

bar; after which they will return to their former position, thus having a motion similar to that of the common pendulum. This arrangement is called a *torsion pendulum*. The motion will be the same for one weight as for two, but when the bar DC is vertical, the arms CA_1 and CA_2 should equal each other, and the weight A_1 equal A_2.

141. *To find the force necessary to twist the rod DC through a given angle.*

Let $F =$ the force at A_2 perpendicular to the arm CA_2;
$a = CA_2 = CA_1$; $l = DC$;
$\alpha = A_1CB_1$; $I =$ the polar moment of inertia of a tran verse section of the bar DC; and
$G =$ the coefficient of the elastic resistance to torsion.

The moment of the twisting force, F, will be

$$Fa;$$

and the moment of the elastic resistance will be (see *Resistance of Materials*, 2d Ed., p. 206),

$$GI\frac{\alpha}{l};$$

hence, we have

$$Fa = GI\frac{\alpha}{l};$$

$$\therefore F = GI\frac{\alpha}{al}.$$

The weights A_1 and A_2 are not involved in this problem.

If the angle be measured from some fixed line making an angle ϕ with the neutral position of A_1A_2, then instead of α we would have $\alpha_1 - \phi$, and the last equation becomes

$$\frac{Fa}{I} = \frac{G}{l}(\alpha_1 - \phi).$$

If the force be reversed, it will twist the bar in the opposite

direction, making an angle a_2 with the fixed line of reference, and we would have

$$\frac{Fa}{I} = \frac{G}{l}(\phi - a_2).$$

Adding these equations gives

$$2\frac{Fa}{I} = \frac{G}{l}(a_1 - a_2).$$

142. *To find the time of an oscillation.*

The bar CD having been twisted by moving the bar from its normal position A_1A_2 into the position B_1B_2, and then left to itself, it is required to find the time of moving to the other extreme position B_3B_4. We will neglect the mass of the rod A_1A_2, and that of the bar DC, and thus simplify the solution, and secure an approximate result.

Let I_A = the moment of inertia of *one* of the bodies, A_1, or A_2, in reference to CD as an axis,

θ = a variable angle measured from the neutral position, A_1A_2; and,

considering Fa as a variable moment, producing the variable angle θ, we have from the second of equations (157), and the value of Fa from the first equation of the preceding Article,

$$-2I_A\frac{d^2\theta}{dt^2} = GI\frac{\theta}{l}.$$

Multiply by $d\theta$ and integrate, and observing that for $\theta = a$ the angular velocity is zero, we find

$$\frac{d\theta^2}{dt^2} = \frac{GI}{2I_Al}(a^2 - \theta^2);$$

hence

$$dt = \sqrt{\frac{2I_Al}{GI}}\frac{d\theta}{\sqrt{a^2-\theta^2}}.$$

Integrating again gives

$$t = \sqrt{\frac{2I_Al}{GI}}\sin^{-1}\frac{\theta}{a},$$

which, between the limits of 0 and a, gives

$$t = \tfrac{1}{2}\pi\sqrt{\frac{2I_A l}{GI}};$$

which is the time of half an oscillation; hence the time for a full oscillation, or the time of movement from B_1 to B_3, will be

$$2t = T = \pi\sqrt{\frac{2I_A l}{GI}};$$

which is the time required. The times are isochronous and independent of the amplitude.

The value of G may be eliminated by substituting its value taken from the last equation of the preceding article. Making the substitution, we find

$$T = \pi\sqrt{\frac{I_A(a_1 - a_2)}{Fa}};$$

from which l and I have also disappeared. From this we find

$$Fa = I_A(a_1 - a_2)\left(\frac{\pi}{T}\right)^2$$

143. *To find the density of the earth.*

The plan of determining the density of the earth by means of a torsion rod was first suggested by the Rev. John Mitchel. He died before he was able to make the experiment, but the plan was executed by Mr. Cavendish, who published the result in the *Phil. Trans.* for 1798. Subsequent to 1837, Mr. Bailey, at the request of the Astronomical Society (England), made a new determination of the result. He made upwards of 2,000 experiments with balls of different weights and sizes, and suspended in a variety of ways, a full account of which is given in the *Memoirs of the Astronomical Society*, Vol. xiv. We give here only some of the more prominent features of the experiment.

The torsion rod DC was very small, so that it could be easily twisted. Two small balls, A_1, A_2, were suspended from the

torsion rod by a light cross-bar. Two large balls, E_1 E_2, were placed on a plank which turned about a point O directly under C, and the whole so arranged that the centres of gravity of the four balls were in the same horizontal plane. The apparatus

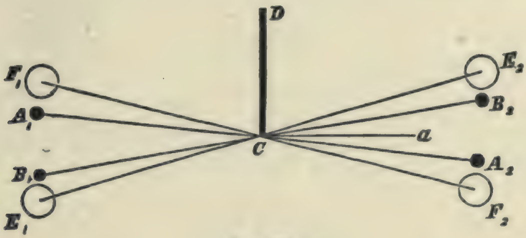


FIG. 128.

was inclosed in a small room so as to exclude currents of air, and the weights E_1 and E_2 were moved into the desired positions from the outside of the room by means of mechanism extending into the room.

The weights E_1 and E_2 were first placed nearly at right angles with the rod $A_1\,A_2$, when the latter would assume some neutral position as Ca. The balls E_1E_2 were then brought quite near to the small ones, B_1, B_2, when the attraction of the former drew the latter from their neutral position, and they oscillated about some position of equilibrium as $B_1\,B_2$. The angle aCB_2 was observed, and also the time of the oscillation about the position CB_2.

The balls were then changed to the position F_1F_2, making the angle aCF_2 as nearly equal as possible to aCE_2; but it was found that the line Ca did not always bisect the angle E_2CF_2, but the mean of many readings was taken as the most probable value. The angle E_2CF_2 will be $a_1 - a_2$, given in the preceding Article.

It is proved, by the law of attraction, that the attraction of a homogeneous sphere is the same as if its entire mass were concentrated at the centre of the mass, and varies inversely as the square of the distance from the centre.

Let $M =$ the mass of one of the large balls;

$m =$ " " " " " " small balls;

$D =$ the distance between their centres; and

μ = the attraction of a sphere whose mass is unity upon another unit when the distance between their centres is unity;

then the force of attraction of the mass M upon m will be

$$\mu \frac{Mm}{D^2};$$

and this is the value of F in the last equation of the preceding Article. But there being two balls in this case, the *moment* of this attractive force will be

$$2\mu \frac{Mm}{D^2} a;$$

which (by neglecting the attraction of the large ball and plank upon the rod CB_2, and of the plank upon the small balls), equals the second member of the last equation of the preceding Article. Hence,

$$2\mu \frac{Mm}{D^2} a = I_A (a_1 - a_2) \left(\frac{\pi}{T}\right)^2$$

Let E be the mass of the earth, R its radius, and g the force of gravity, then

$$g = \mu \frac{E}{R^2}.*$$

Eliminating μ, and making $I_A = m(a^2 + \frac{2}{5} r^2)$, we have

$$\frac{M}{E} = (a^2 + \tfrac{2}{5} r^2)(a_1 - a_2) \frac{D^2}{2R^2 ga} \left(\frac{\pi}{T}\right)^2$$

The density of the earth is thus reduced to the determination of the $a_1 - a_2$ between its two positions of equilibrium when

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* In Bailey's experiments, the value used was

$$g = \mu \frac{E}{R} \left[1 - 2\epsilon + (\tfrac{5}{2} m - \epsilon) \cos^2\lambda\right];$$

in which ϵ is the ellipticity of the earth, m the ratio of the centrifugal force at the equator to equatorial gravity, and λ the latitude of the place.

under the action of the masses in their alternate positions, and the time T of oscillation of the torsion rod. To observe these, a small mirror was attached to the rod at C', with its plane nearly perpendicular to the rod. A scale was engraved on a vertical plate at a distance of 108 inches from the mirror, and the image of the scale formed by reflection on the mirror was viewed in a telescope placed just over the scale. In this way an angle of one or two seconds could be read.

The final result was that the mean density of the earth is 5.6747 times that of distilled water at its maximum density.

144. PROBLEM. *If the earth were a homogeneous sphere, at what point in the radius must it be struck, and what momentum must it receive, that it shall have a velocity of translation of V and of rotation of ω?*

Let $M =$ the mass of the sphere,

$R =$ its radius,

$k_1 = \sqrt{\frac{2}{5}}R =$ the principal radius of gyration. (See Example 4, page 174), and

$a =$ the distance from the centre to the point where the impulse is applied.

The momentum must be

$$MV,$$

wherever the blow is applied. The moment of an impulse being the same as the moment of the momentum, we have, according to equation (155),

$$\omega = \frac{\textit{moment of the momentum}}{\textit{moment of inertia}};$$

$$= \frac{MV.a}{Mk_1^2};$$

$$= \frac{Va}{\frac{2}{5}R^2};$$

$$\therefore a = \tfrac{2}{5}R^2\frac{\omega}{V}.$$

The angular velocity of the earth per hour, is

$$\frac{2\pi}{24};$$

and the linear velocity in the orbit is

68,000 miles per hour nearly;

$$\therefore\ a = \frac{\pi}{2,040,000} R^2.$$

Letting $R = 4,000$ miles, we have

$$a = 24 \text{ miles nearly}.$$

145. Problem. *A homogeneous disc has a motion of translation and of rotation entirely in its own plane, when suddenly any point in the disc becomes fixed; required the angular velocity about the fixed point.*

Let $V =$ the velocity of translation of the centre of the disc;
$\omega =$ the angular velocity about the centre;
$p =$ the perpendicular distance between the fixed point and the line of motion of the centre at the instant that the point becomes fixed;
$r =$ the distance between the fixed point and the centre of the disc;
$k_1 =$ the principal radius of gyration; and
$\omega_1 =$ the angular velocity about the fixed point.

Then

$$\omega_1 = \frac{\textit{moment of the momentum}}{\textit{moment of inertia}}.$$

$$= \frac{Mk_1^2\omega + MVp}{M(k_1^2 + r^2)};$$

$$= \frac{k_1^2\omega + Vp}{k_1^2 + r^2}.$$

If $V = 0$, we have

$$\omega_1 = \frac{k_1^2\omega}{k_1^2 + r^2}.$$

If the centre becomes fixed, we have $p = 0$, and $r = 0$, and

$$\omega_1 = \omega.$$

146. Problem. *A sphere whose radius is R has an angular velocity ω, and gradually contracts until its radius is r; required the final angular velocity.*

We will assume that the body remains homogeneous throughout, and that there is no change of temperature, and that the change of volume is due simply to the mutual attraction of the particles for each other, which is supposed to draw them towards the centre. Then will the moment of the momentum be constant.

Let ω_1 be the required angular velocity; then we have from equation (155)

$$Qa = \omega M.\tfrac{2}{5} R^2 = \omega_1 M_1.\tfrac{2}{5} r^2;$$

$$\therefore \omega_1 = \frac{R^2}{r^2}\omega.$$

Problem.—A spherical homogeneous mass m, radius r, contracts by the mutual attraction of its particles to a radius nr; if the work thus expended be suddenly changed into heat, how many degrees F. will the temperature of the mass be increased, its specific heat being s and the heat uniformly disseminated?

Consider the earth as a homogeneous sphere, mass M, radius R, δ its density compared with water, g the acceleration on the surface of the earth due to gravity, g' that on the surface of m. Then, since attractions are directly as the masses and inversely as the square of the radii, we have $g' = \frac{m}{M}\cdot\frac{R^2}{r^2}g$. Initially, the force within the sphere varies directly as the distance ρ from the centre, but during contractions as the inverse squares; hence the acceleration at a distance x of a particle originally at ρ will be $\frac{m}{M}\cdot\frac{R^2}{r^2}g\cdot\frac{\rho}{r}\cdot\frac{\rho^2}{x^2}$. The mass of a spherical shell, radius ρ, is $dm = (m \div \frac{4}{3}\pi r^3)\,4\pi\rho^2 d\rho = \frac{3m}{r^3}\rho^2 d\rho$. The force equals the mass into the acceleration, and this by dx will be an element of work; hence the total work will be

$$-\frac{3m^2}{M}\cdot\frac{R^2}{r^6}g\int_0^r\int_\rho^{n\rho}\frac{\rho^5}{x^2}\,d\rho\,dx = \frac{3}{5}\frac{m^2}{M}\frac{R^2}{r}\frac{1-n}{n}g.$$

Dividing by J (Joule's mechanical equivalent of heat), also by the weight of a sphere of water equal in volume to that of the given sphere, or $\frac{m}{M}\cdot\frac{Mg}{\delta}$, and also by the specific heat of the body, we finally have $T = \frac{3mR^2\delta}{5sJMr}$.

—Problem by the Author in *Mathematical Visitor*, July, 1880.

CHAPTER XI.

GENERAL EQUATIONS OF MOTION.

147. D'ALEMBERT'S PRINCIPLE.—A body is a collection of material particles held together by a force exerted by each particle upon the others. Having deduced the laws of action for forces acting upon a single particle, the direct process for determining the effect of forces upon a body of finite size, appears to be to consider all the forces which act upon each particle separately, including the mutual actions and reactions of the particles, thus establishing equations for each particle of the body, and then to eliminate the terms involving the actions and reactions. But the latter are generally unknown. Various expedients were resorted to by the ancient mathematicians to reach the resulting equations, but the principle announced by D'Alembert greatly simplified the operations, and in many cases reduced the establishment of the equations to Statical principles. See Whewell's History of the Inductive Sciences, Vol. I., p. 365.

The forces which produce the motion of a body may be applied at only a few points, and yet produce motion in every particle of it. These forces are called *impressed forces.* If we consider the particles as separated from each other, and forces applied to them which will produce the same motion that they had when in the body, the latter forces are called *effective forces.* The *effective forces* will then produce the same effect upon the body as the *impressed forces.* D'Alembert's principle consists in this, that *if a system of forces, equal and opposite to the* EFFECTIVE FORCES, *act upon a body, they will be in equilibrium with the* IMPRESSED FORCES.

In this principle no assumption has been made in regard to the character of the mutual actions and reactions between the particles, and hence it is applicable to flexible bodies and fluids, as well as to solids. It is equivalent to assuming that the forces

within a body constitute a system which are in equilibrium among themselves.

148. To FIND THE GENERAL EQUATIONS OF MOTION OF A BODY. Let x_1, y_1, z_1, be the coördinates of a particle whose mass is m_1; X_1, Y_1, Z_1, the impressed forces parallel to the respective axes acting upon the particle; and a similar notation for the other particles. The measure of the accelerating force parallel to the axis of x will be

$$m_1 \frac{d^2x_1}{dt^2},$$

and if a force equal and opposite to this act upon the particle there will be equilibrium; hence we have

$$X_1 - m_1 \frac{d^2x_1}{dt^2} = 0;\ \ Y_1 - m_1 \frac{d^2y_1}{dt^2} = 0;\ \ Z_1 - m_1 \frac{d^2z_1}{dt^2} = 0.$$

Similarly,

$$X_2 - m_2 \frac{d^2x_2}{dt^2} = 0;\ \ Y_2 - m_2 \frac{d^2y_2}{dt^2} = 0;\ \ Z_2 - m_2 \frac{d^2z_2}{dt^2} = 0;$$

and similar expressions for all the other particles of the body. But if ΣX, ΣY, ΣZ, be the sum of the respective axial components of the *impressed forces*, then

$$\Sigma X = X_1 + X_2 + X_3 + \text{etc.};$$

and similarly for the others. Hence, if m be the mass of any particle whose coördinates are x, y, z, at the time t, we have according to D'Alembert's principle

$$\Sigma X - \Sigma m \frac{d^2x}{dt^2} = 0;$$

and similarly for the others.

Taking the moments of the forces in reference to the axis of x, we have, in precisely the same way, the equation

$$\Sigma Zy - \Sigma Yz - \Sigma\left(my\frac{d^2z}{dt^2} - mz\frac{d^2y}{dt^2}\right) = 0;$$

and similarly for the other axis. Hence, we have the following six equations:—

$$\left.\begin{aligned}
\Sigma X - \Sigma m \frac{d^2x}{dt^2} &= 0; \\
\Sigma Y - \Sigma m \frac{d^2y}{dt^2} &= 0; \\
\Sigma Z - \Sigma m \frac{d^2z}{dt^2} &= 0;
\end{aligned}\right\} \qquad (164)$$

$$\left.\begin{aligned}
\Sigma(Zy - Yz) - \Sigma\left(my\frac{d^2z}{dt^2} - mz\frac{d^2y}{dt^2}\right) &= 0; \\
\Sigma(Xz - Zx) - \Sigma\left(mz\frac{d^2x}{dt^2} - mx\frac{d^2z}{dt^2}\right) &= 0; \\
\Sigma(Yx - Xy) - \Sigma\left(mx\frac{d^2y}{dt^2} - my\frac{d^2x}{dt^2}\right) &= 0.
\end{aligned}\right\} \qquad (165)$$

Let x_1, y_1, z_1, be the coördinates of any particle of the body referred to a movable system whose origin remains at the centre of the mass, and whose axes are parallel to the fixed axes, and

$\bar{x}, \bar{y}, \bar{z}$, the coördinates of the centre of the mass referred to the fixed origin;

then we have

$$x = \bar{x} + x_1;$$

$$\Sigma mx = \Sigma m\bar{x} + \Sigma mx_1;$$

$$\Sigma m\frac{d^2x}{dt^2} = \Sigma m\frac{d^2\bar{x}}{dt^2} + \Sigma m\frac{d^2x_1}{dt^2}.$$

But the origin of the movable system being at the centre of the mass, we have, from equations (71*a*) or (84*a*),

$$\Sigma mx_1 = 0;$$

$$\therefore\ \Sigma m\frac{d^2x_1}{dt^2} = 0;$$

and the last of the preceding equations becomes

$$\Sigma m\frac{d^2x}{dt^2} = \Sigma m\frac{d^2\bar{x}}{dt^2}$$

$$= \frac{d^2\bar{x}}{dt^2}\Sigma m,$$

since $\frac{d^2\bar{x}}{dt^2}$ is a common factor to all the particles m; but $\Sigma m = M$;

$$\therefore \Sigma m \frac{d^2x}{dt^2} = M \frac{d^2\bar{x}}{dt^2},$$

and similarly for the others. Hence, equations (164) become

$$\left.\begin{aligned} M\frac{d^2\bar{x}}{dt^2} &= \Sigma X; \\ M\frac{d^2\bar{y}}{dt^2} &= \Sigma Y; \\ M\frac{d^2\bar{z}}{dt^2} &= \Sigma Z. \end{aligned}\right\} \qquad (166)$$

Similarly, in equations (165), we have

$$\Sigma my \frac{d^2z}{dt^2} =$$

$$\Sigma\{m(\bar{y} + y_1)\left(\frac{d^2\bar{z}}{dt^2} + \frac{d^2z_1}{dt^2}\right) =$$

$$\Sigma m\bar{y}\frac{d^2\bar{z}}{dt^2} + \Sigma m\bar{y}\frac{d^2z_1}{dt^2} + \Sigma my_1\frac{d^2\bar{z}}{dt^2} + \Sigma my_1\frac{d^2z_1}{dt^2}.$$

But $\bar{y}$ and $\bar{z}$ are common factors in their respective terms, therefore the expression becomes

$$\bar{y}\frac{d^2\bar{z}}{dt^2}\Sigma m + \bar{y}\,\Sigma m\frac{d^2z_1}{dt^2} + \frac{d^2\bar{z}}{dt^2}\Sigma my_1 + \Sigma my_1\frac{d^2z_1}{dt^2};$$

but,

$$\Sigma my_1 = 0; \qquad \Sigma m\frac{d^2z_1}{dt^2} = 0;$$

hence, we finally have

$$\Sigma my\frac{d^2z}{dt^2} = M\bar{y}\frac{d^2\bar{z}}{dt^2} + \Sigma my_1\frac{d^2z_1}{dt^2};$$

and similarly for other terms. In this way the first of equations (165) becomes

$$\Sigma Zy_1 - \Sigma Yz_1 + \Sigma Z\bar{y} - \Sigma Y\bar{z} - \Sigma\left(my_1\frac{d^2z_1}{dt^2} - mz_1\frac{d^2y_1}{dt^2}\right) - M\left(\bar{y}\frac{d^2\bar{z}}{dt^2} - \bar{z}\frac{d^2\bar{y}}{dt^2}\right) = 0.$$

Multiply the third of equations (166) by $\bar{y}$, the second by $\bar{z}$, subtract the latter from the former, and we have

$$M\bar{y}\frac{d^2\bar{z}}{dt^2} - M\bar{z}\frac{d^2\bar{y}}{dt^2} = \Sigma Z\bar{y} - \Sigma Y\bar{z};$$

which, substituted in the preceding equation, gives

$$\Sigma\left(my_1\frac{d^2z_1}{dt^2} - mz_1\frac{d^2y_1}{dt^2}\right) = \Sigma Zy_1 - \Sigma Yz_1.$$

Dropping Σ before X, Y, Z, and letting those letters represent the *total* axial components upon the *entire* body, and $Zy_1 - Yz_1$, etc., the resultant moments of the applied forces, we have the six following equations:

$$\left.\begin{aligned} M\frac{d^2\bar{x}}{dt^2} &= X; \\ M\frac{d^2\bar{y}}{dt^2} &= Y; \\ M\frac{d^2\bar{z}}{dt^2} &= Z; \end{aligned}\right\} \qquad (167)$$

$$\left.\begin{aligned} \Sigma\left(my_1\frac{d^2z_1}{dt^2} - mz_1\frac{d^2y_1}{dt^2}\right) &= Zy_1 - Yz_1; \\ \Sigma\left(mz_1\frac{d^2x_1}{dt^2} - mx_1\frac{d^2z_1}{dt^2}\right) &= Xz_1 - Zx_1; \\ \Sigma\left(mx_1\frac{d^2y_1}{dt^2} - my_1\frac{d^2x_1}{dt^2}\right) &= Yx_1 - Xy_1. \end{aligned}\right\} \qquad (168)$$

Equations (167) do not contain the coördinates of the point of application of the forces, hence, *the motion of translation of the centre of a mass is independent of the point of application of the force or forces; or, in other words, it is independent of the rotation of the mass.*

Equations (168) do not contain the coördinates of the centre of the mass, and being the equations for rotation, show that *the rotation of a mass is independent of the translation of its centre.*

These equations are sufficient for determining all the circumstances of motion of a free solid. In their further use the dashes and subscripts will be omitted.

149. If X, Y, Z, are zero, we have

$$\int \frac{d^2x}{dt^2} = \frac{dx}{dt} + C_1 = 0;$$

and similarly for the others. Transposing, squaring, adding and extracting the square root, give

$$v = \sqrt{\frac{dx^2 + dy^2 + dz^2}{dt^2}} = \sqrt{C_1^2 + C_2^2 + C_3}; \quad (169)$$

which, being constant, shows that *the motion of the centre of the mass is rectilinear and uniform.*

This is *the general principle of the* CONSERVATION OF THE CENTRE OF GRAVITY.

CONSERVATION OF AREAS.

150. The expression,

$$xdy - ydx,$$

is, according to Article 112, twice the sectoral area passed over by the radius vector of the body in an instant of time. Hence, if

$$\Sigma(mxdy - mydx) = dA_1;$$

differentiating, we find

$$\Sigma\left(mx\frac{d^2y}{dt^2} - my\frac{d^2x}{dt^2}\right) = \frac{d^2A_1}{dt^2}.$$

If there are no accelerating forces $m\dfrac{d^2x}{dt^2} = 0$; and similarly for the others; hence

$$\frac{d^2A_1}{dt^2} = 0; \qquad \frac{d^2A_2}{dt^2} = 0; \qquad \frac{d^2A_3}{dt^2} = 0;$$

$$\therefore A_1 = c_1t; \qquad A_2 = c_2t; \qquad A_3 = c_3t; \qquad (170)$$

the initial values being zero. These are the projections on the coördinate planes of the areas swept over by the radius vector of the body. They establish the principle of the CONSERVATION OF AREAS. That is,

In any system of bodies, moving without accelerating forces and having only mutual actions upon each other, the projections on any plane of the areas swept over by the radius vector are proportional to the times.

CONSERVATION OF ENERGY.

151. Multiply equations (167) by dx, dy, dz, respectively, add and integrate, and we have

$$Mv^2 - Mv_0^2 = 2\int(Xdx + Ydy + Zdz);$$

and for a system of bodies, we have

$$\tfrac{1}{2}\Sigma(Mv^2) - \tfrac{1}{2}\Sigma(Mv_0^2) = \Sigma\int(Xdx + Ydy + Zdz). \quad (171)$$

The second member is integrable when the forces are directed towards fixed centres and is a function of the distances between them.

Let a, b, c be the coördinates of one centre,
a_1, b_1, c_1, of another, etc.;
x, y, z, the coördinates of the particle m;
r, r_1, etc., be the distances of the particle from the respective centres;
F, F_1, etc., be the forces directed towards the respective centres;

then, resolving the forces parallel to the axes, we have

$$X = F\cos a + F_1 \cos a + \text{etc.}$$
$$= F\frac{a-x}{r} + F_1\frac{a_1 - x}{r} + \text{etc.};$$

$$Y = F\frac{b-y}{r} + F_1\frac{b_1-y}{r} + \text{etc.};$$

$$Z = F\frac{c-z}{r} + F_1\frac{c_1-z}{r} + \text{etc.}$$

Multiplying by dx, dy, dz, respectively, and adding, we have

$$Xdx + Ydy + Zdz = F\left\{\frac{a-x}{r}dx + \frac{b-y}{r}dy + \frac{c-z}{r}dz\right\}$$
$$+ F_1\left\{\frac{a_1-x}{r_1}dx + \frac{b_1-y}{r_1}dy + \frac{c_1-z}{r_1}dz\right\}$$
$$+ \text{etc.}$$

But $$r^2 = (a-x)^2 + (b-y)^2 + (c-z)^2; \qquad (172)$$

and by differentiating, we find

$$dr = -\frac{a-x}{r}dx - \frac{b-y}{r}dy - \frac{c-z}{r}dz;$$

similarly,

$$dr_1 = -\frac{a_1-x}{r_1}dx - \frac{b_1-y}{r_1}dy - \frac{c_1-z}{r_1}dz;$$
$$\text{etc.,} \qquad \text{etc.,} \qquad \text{etc.}$$

These substituted above, give

$$Xdx + Ydy + Zdz = -Fdr - F_1dr_1 - F_2dr_2 - \text{etc.}$$

Therefore, if F, etc., is a function of r, etc., and μ, μ_1, etc., the intensities of the respective forces at a distance unity from the respective centres, or

$$F = \mu\phi(r);$$
$$F_1 = \mu_1\phi(r_1);$$
$$\text{etc.,} \quad \text{etc.};$$

the second member, and hence the first, will be integrable.

In nature, if a particle m attracts a particle m_1, the particle m_1 will attract m, each being a centre of force in reference to the other, and *both* centres will be *movable* in reference to a fixed origin. But one centre may be considered fixed in reference to the other, and consequently the proposition remains true for this case.

The second member of equation (171) being integrated between the limits x, y, z and x_0, y_0, z_0, we have

$$\tfrac{1}{2}\Sigma(Mv^2) - \tfrac{1}{2}\Sigma(Mv_0^2) = \Sigma\mu\varphi(x, y, z) - \Sigma\mu\varphi(x_0, y_0, z_0). \quad (173)$$

Hence, *the gain or loss of energy of a system, subject to forces directed towards fixed centres and which are functions of the distances from those centres, is independent of the path*

described by the bodies, and depends only upon the position left and arrived at by the bodies, and the intensities of the forces at a unit's distance from the respective centres.

Therefore, when the system returns to the initial position, or to a condition equivalent to the original one, the vis viva will be the same.

From equation (171) we have

The gain or loss of energy in passing from one position to another equals the work done by the impressed forces.

Let W_0 = the work done by the impressed forces in passing from some definite point $(x_0, y_0, z_0,)$ to another definite point $(x_1, y_1, z_1,)$; and

W = the work done by the same forces in passing from the first point to any point $(x, y, z,)$;

then from equation (171), we have

$$\tfrac{1}{2}\Sigma(Mv_0^2) - \tfrac{1}{2}\Sigma(Mv_1^2) = W_0;$$
$$\tfrac{1}{2}\Sigma(Mv_0^2) - \tfrac{1}{2}\Sigma(Mv^2) = W;$$

and subtracting the latter from the former, gives

$$\tfrac{1}{2}\Sigma(Mv^2) - \tfrac{1}{2}\Sigma(Mv_1^2) = W_0 - W;$$

or, by transposing,

$$\tfrac{1}{2}\Sigma(Mv^2) + W = \tfrac{1}{2}\Sigma(Mv_1^2) + W_0;$$

in which the second member is constant.

The first term of the first member is the kinetic energy which the system has at any point of the path, and the second term is the work which has been done by the forces upon the body and has become *latent*, or potential; hence, in such a system *the sum of the kinetic and potential energies is constant.*

This is *the principle of* THE CONSERVATION OF ENERGY in theoretical mechanics. This term has been extended so as to include the principle of the *transmutation of energy* as established by physical science.

If a portion of the universe, as the Solar System for instance, be separated from all external forces, the sum of the kinetic and potential energies will remain constant, so that if the kinetic energy diminishes, the potential increases and the

converse. If external forces act, the potential and kinetic energies may both be increased.

To be more specific, suppose that the earth and sun constitute the system, the sun being considered the centre of the force. The velocity of the earth will be greatest when nearest the sun, and will diminish as it recedes from it. While receding, the amount of work done *against* the attractive force of the sun will be

$$\tfrac{1}{2}Mv^2 - \tfrac{1}{2}Mv_1^2 = -W;$$

in which M is the mass of the earth, v_1 the maximum velocity, and v the velocity at any point. The second member is negative because the first member is.

When the earth is approaching the sun the velocity is increased and the living force is restored, and the kinetic added to the potential energy is constant.

Again, if a body whose weight is W be raised a height h, the work which has been done to raise it to that point is Wh, and in that *position* its potential energy is Wh. If it falls freely through that height it will acquire a velocity $v = \sqrt{2gh}$,

$$\therefore\ Wh = \frac{W}{2g}v^2 = \tfrac{1}{2}Mv^2$$

which is the kinetic energy.

If the same body fall through a portion of the height, say h_1, its kinetic energy will be $Wh_1 = \frac{1}{2}Mv_1^2$, and the work which is still due to its *position* is $W(h-h_1)$, which, at that instant, is *inert* or potential.

It is found, however, that in the use of machines or other devices, by which work is transmitted from one body to another, all the work stored in a moving body cannot be utilized. Thus, in the impact of non-elastic bodies there is always a loss of living force. (See Article 32.)

This, so far as theoretical mechanics is concerned, is a loss, and is treated as such, and until modern Physical Science established *the correlation of forces* it was supposed to be entirely lost. But we know that in the case of impact heat is developed, and Joule determined a definite relation between the quantity of heat and the work necessary to produce it, and called the result *the mechanical equivalent of heat.* Further

investigations show that in every case of a supposed loss of energy, it may be accounted for in a general way by the appearance of energy in some other form. It is impossible to trace the transformation of energy as it appears in mechanical action, friction, heat, light, electricity, megnetism, etc., and prove by direct measurements that the sum total at every instant and with every transformation remains rigidly constant; but by means of careful observations and measurements nearly all the energy in a variety of cases has been traced from one mode of action to another, and the small fraction which was apparently lost could be accounted for by the imperfections of the apparatus, or in some other satisfactory manner; until at last the principle of *the conservation of energy* is recognized to be a law as universal as that of the law of gravitation. The *exact nature* of molecular energy which manifests itself in heat, chemical affinity, etc., are unknown, but, according to the general law, all energy whether molecular or of finite masses, is either *kinetic* or *potential.*

COMPOSITION OF ROTARY MOTIONS.

152. Take the origin of coördinates at the fixed point when there is one, or at any point of the axis of rotation when the body is free. Conceive three cones having a common vertex at the origin of coordinates and each tangent respectively to the coördinate planes; then will the angular velocity of that element of the cone which for the instant is in contact with the plane xy, considered as rotating about the axis of z, be *defined* as the angular velocity of *the body* about that axis; and similarly for the other axes. Let ω_x, ω_y, ω_z, be the angular velocities of *the body* about the respective axes x, y, z. If ca, considered as infinitesimal, be the actual velocity of a particle in the plane xy, positive from x towards y, $\rho = Oc$, $ca = \rho\omega_z$; then

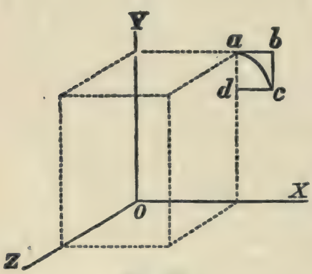

FIG. 127.

$$cd = \rho\omega_z \cos(180° - acd) = -\omega_z \cdot \rho \sin XOc = -\omega_z y,$$
$$da = \rho\omega_z \sin(180° - acd) = \omega_z \cdot \rho \cos XOc = \omega_z x.$$

Similarly in regard to the axis of y, we have

$$-\omega_y x, \qquad \omega_y z,$$

and in regard to x,

$$-\omega_x z, \qquad \omega_x y.$$

When all these rotations take place at the same time, we have, by adding the corresponding velocities, the several velocities along the axes

$$\left.\begin{aligned} \frac{dx}{dt} &= \omega_y z - \omega_z y; \\ \frac{dy}{dt} &= \omega_z x - \omega_x z; \\ \frac{dz}{dt} &= \omega_x y - \omega_y x. \end{aligned}\right\} \tag{174}$$

The particles on the instantaneous axis have no velocity in reference to the movable origin, hence

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = 0;$$

$$\therefore\ \omega_y z - \omega_z y = 0, \quad \omega_z x - \omega_x z = 0, \quad \omega_x y - \omega_y x = 0; \tag{175}$$

which are the equations of a straight line through the origin, and are the equations of the instantaneous axis. Let a, β, γ, be the angles which it makes with the axes x, y, z, respectively, then (*Anal. Geom.*),

$$\cos a = \frac{\omega_x}{\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}}; \quad \cos \beta = \frac{\omega_y}{\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}};$$

$$\cos \gamma = \frac{\omega_z}{\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}}.$$

To determine the angular velocity of the body, take any point in a plane perpendicular to the instantaneous axis. Let the point be on the axis of x, and from it erect a perpendicular to the instantaneous axis, and we have

$$p = x \sin a = x\sqrt{1 - \cos^2 a} = \sqrt{\frac{\omega_y^2 + \omega_z^2}{\omega_x^2 + \omega_y^2 + \omega_z^2}}\,x.$$

For this point $y = 0$ and $z = 0$ in equations (174), and we find for the actual velocity,

$$V = \frac{\sqrt{dx^2 + dy^2 + dz^2}}{dt} = x\sqrt{\omega_y^2 + \omega_z^2};$$

and hence

$$\omega = \frac{V}{p} = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}; \qquad (176)$$

which represents the diagonal of a rectangular parallelopipedon, of which the sides are ω_x, ω_y, ω_z.

153. Moments *of rotation of the centre of the mass about the fixed axes.*

Multiply the second of equations (167) by $\bar{z}$, the third by $\bar{y}$, subtract the former from the latter, and we have

$$M\left(\bar{y}\frac{d^2\bar{z}}{dt^2} - \bar{z}\frac{d^2\bar{y}}{dt^2}\right) = Z\bar{y} - Y\bar{z}.$$

Treating the equations two and two in this manner, dropping the dashes, and substituting L_1, M_1, N_1, for the second members, we have

$$\left.\begin{aligned} M\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) &= L_1; \\ M\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) &= M_1; \\ M\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) &= N_1. \end{aligned}\right\} \qquad (177)$$

These equations are of the same *form* as equations (168).

MOTION OF A BODY DURING IMPACT.

154. *Motion of the centre of the masses.* The second members of equations (167) are the accelerating forces. If any two of the bodies collide, they being free in other respects, the action of one body upon the other is equal and opposite to that of the latter upon the former; hence, in regard to the system they neutralize each other, and the motion of the centre of the masses will be unaffected by the collision. If there are no

accelerating forces the velocity of the centre will be uniform and in a straight line, as shown in Article 140.

To find the velocity of the bodies after impact requires a knowledge of their physical constitution. See Articles 28 and 29.

155. *The motion of rotation* of the *centre* of the entire mass about the origin will also be unaffected by the collision, when the bodies are acted upon by accelerating forces; for, the moments of the forces due to the collision will neutralize each other, and the second members of equations (177) will contain only the applied forces.

This would be illustrated by the impact of two asteroids, or in the bursting of a primary planet.

But the rotation produced in each body about the centre of its mass depends upon the moments of the forces applied to the body, and hence, upon the moment of the momentum produced by the impact.

CONSTRAINED MOTION.

156. *General equations of rotation about a fixed point.*

Take the origin of coördinates at the fixed point. For this case equations (164) vanish. Substitute in (165), the values of $\frac{d^2x}{dt^2}$, etc., obtained from (174), and reduce. We have

$$\frac{d^2x}{dt^2} = z\frac{d\omega_y}{dt} - y\frac{d\omega_z}{dt} + \omega_y(\omega_x y - \omega_y x) - \omega_z(\omega_z x - \omega_x z);$$

and similarly for the others.

Let L, M, N, be substituted for the last term respectively of equations (165), and substituting the above values in the last of these equations, we find

$$\left.\begin{array}{c} \frac{d\omega_z}{dt}\Sigma m(x^2+y^2) + \omega_x\omega_y\Sigma m(x^2-y^2) \\ -(\omega_x^2-\omega_y^2)\Sigma mxy - \left(\frac{d\omega_y}{dt}+\omega_x\omega_z\right)\Sigma myz \\ +\left(\omega_y\omega_z - \frac{d\omega_x}{dt}\right)\Sigma mxz. \end{array}\right\} = N. \quad (178)$$

The other two equations may be treated in the same manner. But they are too complicated to be of use. Since the position of the axes is arbitrary, let them be so chosen that

$$\Sigma mxy = 0, \qquad \Sigma mxz = 0, \qquad \Sigma myz = 0; \qquad (179)$$

in which case the axes are called *principal axes;* and we will show in the next article, that, for every point of a body, there are at least *three* principal axes, each of which is perpendicular to the plane of the other two.

Let x_1, y_1, z_1, be the principal axes, having the same origin as the fixed axes, and

$A = \Sigma m(y_1^2+z_1^2)$, the moment of inertia of the body about x_1;

$B = \Sigma m(z_1^2+x_1^2)$, moment about y_1;

$C = \Sigma m(x_1^2+y_1^2)$, moment about z_1;

also let ω_1, ω_2, ω_3, be the angular velocities about the respective axes x_1, y_1, and z_1, and substituting these several quantities in (178), we have

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + (C - B)\,\omega_2\omega_3 &= L; \\ B\frac{d\omega_2}{dt} + (A - C)\,\omega_3\omega_1 &= M; \\ C\frac{d\omega_3}{dt} + (B - A)\,\omega_1\omega_2 &= N. \end{aligned}\right\} \qquad (180)$$

These are called Euler's Equations.

The origin of coördinates may be taken at the centre of the mass, and as the rotation about that point is the same whether that point be at rest or in motion, as shown at the bottom of page 234, equations (180) are applicable to the rotation of a free body when acted upon by forces in any manner.

PRINCIPAL AXES.

157. *At every point of a body there are at least three principal axes perpendicular to each other.*

When three axes meeting at a point in a body are perpendicular to each other, and so taken that

$$\Sigma mxy = 0, \qquad \Sigma myz = 0, \qquad \Sigma mzx = 0;$$

they are called Principal Axes.

The planes containing the principal axes are called Principal Planes.

The moments of inertia in reference to the principal axes at any point are called the Principal Moments of Inertia for that point.

Let ON be any line drawn through the origin, making angles α, β, γ, with the respective coördinate axes. First find the moment of inertia about the line ON. From any point of the line ON, erect a perpendicular, NP. The coördinates of P will be x, y, z. Hence we have

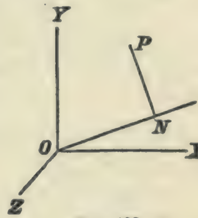

FIG. 128.

$$OP^2 = x^2+y^2+z^2;$$
$$ON = x\cos\alpha + y\cos\beta + z\cos\gamma;$$
$$1 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma.$$

The moment of inertia of the body in reference to ON, will be

$$\begin{aligned}
I &= \Sigma m\,.\,PN^2 = \Sigma m\,(OP^2 - ON^2)\\
&= \Sigma m\,\{(x^2+y^2+z^2 - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2\}\\
&= \Sigma m\,\{(x^2+y^2+z^2)(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - (x\cos\alpha + y\cos\beta + z\cos\gamma)^2\}\\
&= \Sigma m\,(y^2+z^2)\cos^2\alpha + \Sigma m\,(x^2+z^2)\cos^2\beta + \Sigma m\,(x^2+y^2)\cos^2\gamma\\
&\quad - 2\Sigma myz\cos\beta\cos\gamma - 2\Sigma mzx\cos\gamma\cos\alpha - 2\Sigma mxy\cos\alpha\cos\beta\\
&= A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - 2D\cos\beta\cos\gamma - 2E\cos\gamma\cos\alpha - 2F\cos\alpha\cos\beta;
\end{aligned}$$

in which A, B, C, have the values given on page 244, and D, E, F, are written for the corresponding factors of the preceding equation.

This may be illustrated geometrically. Conceive a radius vector, r, to move about in space in such a manner that for all angles α, β, γ, corresponding to those of the line ON, the square of the length shall be inversely proportional to the moment of inertia of the body. Then

$$I = \frac{c}{r^2};$$

in which c is a constant. Hence, the polar equation of the locus is

$$\frac{c}{r^2} = A\cos^2\alpha + B\cos^2\beta + C\cos^2\gamma - 2D\cos\beta\cos\gamma - 2E\cos\gamma\cos\alpha - 2F\cos\alpha\cos\beta.$$

Multiplying by r^2, we have

$$c = Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy;$$

which is the equation of the locus referred to rectangular coördinates, and is a quadric. Since A, B, and C are essentially positive, it is the equation of an ellipsoid, and is called *the momental ellipsoid.* Therefore, the moment of inertia about every line which passes through any point of a body may be represented by the radius vector of a certain ellipsoid. But every ellipsoid has at least three principal diameters, hence every material system has, at every point of it, at least three principal axes.

If the ellipsoid be referred to its principal diameters the coefficients of yz, zx, xy, vanish, and the equation of the ellipsoid becomes

$$c = Ax^2 + By^2 + Cz^2.$$

In many cases the principal diameters may be determined by inspection. Thus, in a sphere every diameter is a principal axis. In an ellipsoid the three axes are principal axes. In all surfaces of revolution, the axis of revolution is a principal axis, and any two lines perpendicular to each other and to the axis of revolution are the other two principal axes.

158. *If a body revolve about one of the principal axes passing through the centre of gravity of the body, that axis will suffer no strain from the centrifugal force.*

Let z be a principal axis, about which the body rotates. The centrifugal force of any particle will be

$$m\omega^2\rho = m\omega^2\sqrt{x^2+y^2};$$

which, resolved parallel to x and y, gives

$$m\omega^2 x, \qquad m\omega^2 y;$$

and the moments of these forces about the axis of z are, for the whole body,

$$\Sigma m\omega^2 xy, \qquad \Sigma m\omega^2 yx;$$

but these, according to equations (179), are zero. If the body be free and revolves about this axis it will continue to revolve about it. For this reason it is called *an axis of permanent rotation.*

If the body be free, and the initial rotation be not about a principal axis, the centrifugal force will cause the *instantaneous* axis to change constantly, and it will never rotate about the *permanent axis.* If, therefore, we observe that a free body revolves about an axis for a short time, we infer that it revolved about it from the beginning of the motion.

RELATION BETWEEN THE AXES x, y, z, FIXED IN SPACE AND THE PRINCIPAL AXES x_1, y_1, z_1, FIXED IN THE BODY.

159. If the body be free, take the origin of coördinates O, Fig. 129, at the centre of the mass; but if there be a fixed point about which rotation is forced to take place, take the origin O at that point. Conceive a sphere, radius unity, having its centre also at O. The line PP' will be the intersection of the planes xy and x_1y_1, and P one of the points where it pierces the surface of the sphere.

Positive angles will be determined in substantially the same manner as positive moments* described in Article 54, page 80; thus, positive rotation will be from $+x$ towards $+y$, from $+y$ towards $+z$, and from $+z$ towards $+x$; and the opposite directions will be negative. In the following figure the rotations are all positive, and the angles or amounts of rotation are represented as less than 90°.

In passing from one system of rectangular axes to another,

* It may be observed that the relations of the axes x, y, and z, in the following figures are not the same as on the preceding pages. Thus, for instance, on pages 70 and 111 the axis of y is vertically upward; while in the following figures z will be in that position. While the author prefers the former arrangement, for the reason that the axes x and y would then retain the same relative position in the printed page as is most common when only two axes are used; yet for the sake of conforming with the usage of most writers, and for greater ease in comparing results, we have concluded to make this change. It is proper to observe that this will cause no change whatever in the preceding analysis, provided the *order* of the letters be observed, to the exclusion of *right* and *left* handed rotation.

the origin being the same, we may proceed as follows: The system may be turned in a positive direction about z as an axis, bringing OX to the position of OP; then rotating it positively about OP as an axis, bringing OZ into the position OZ_1; and finally a positive rotation of the system about OZ_1, bringing OP into the position OX, the final position of OY being OY_1.

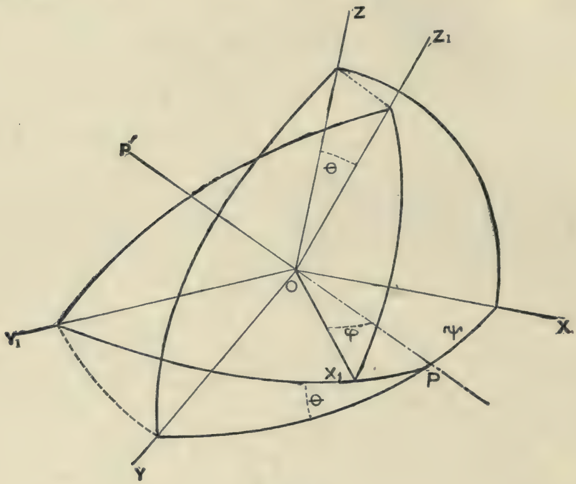

FIG. 129.

Then let

$\theta = ZOZ_1$, being the angle between the axes z and z_1 which is also the angle between the planes xy and $x_1 y_1$;

$\psi = XOP$, being the angle between the axis of x and the line OP;

$\varphi = POX_1$, being the angle between the line OP and the new axis of x_1.

In astronomical language the line OP is called the line of nodes, and PX_1 the right ascension.

Expressing the above steps in the process analytically, we have Euler's method of passing from one system of axes to the other. Thus, let x', y', z', be the position of the new axes at the end of the first step, then we have (*Coördinate Geometry*, Art. 54, or Art. 219)

$$
\begin{aligned}
x &= x' \cos \psi - y' \sin \psi, \\
y &= x' \sin \psi + y' \cos \psi, \\
z &= z'.
\end{aligned}
$$

Next, turn the system in a positive direction about x' as an axis, through an angle θ, and let x'', y'', z'', be the position of the axes at the end of the second step, then

$$
\begin{aligned}
x' &= x'', \\
y' &= y'' \cos \theta - z'' \sin \theta, \\
z' &= y'' \sin \theta + z'' \cos \theta.
\end{aligned}
$$

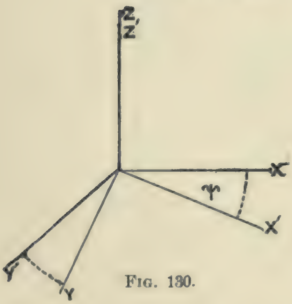

FIG. 130.

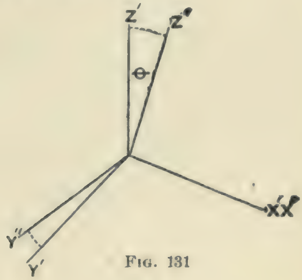

FIG. 131

Finally, turn the system in a positive direction about z' through an angle φ, and let x_1, y_1, z_1, be the final position of the axes, then

$$
\begin{aligned}
x'' &= x_1 \cos \varphi - y_1 \sin \varphi, \\
y'' &= x_1 \sin \varphi + y_1 \cos \varphi, \\
z'' &= z_1.
\end{aligned}
$$

FIG. 132.

Eliminating x', y', z', x'', y'', z'', between these equations gives

$$
\left.
\begin{aligned}
x = {} & (\cos \varphi \cos \psi - \sin \varphi \sin \psi \cos \theta)\, x_1 \\
& + (-\sin \varphi \cos \psi - \cos \varphi \sin \psi \cos \theta)\, y_1 \\
& + \sin \psi \sin \theta \,.\, z_1 \\
y = {} & (\cos \varphi \sin \psi + \sin \varphi \cos \psi \cos \theta)\, x_1 \\
& + (-\sin \varphi \sin \psi + \cos \varphi \cos \psi \cos \theta)\, y_1 \\
& - \cos \psi \sin \theta .\, z_1 \\
z = {} & \sin \varphi \sin \theta \,.\, x_1 + \cos \varphi \sin \theta \,.\, y_1 \\
& + \cos \theta \,.\, z_1
\end{aligned}
\right\} . \qquad (181)
$$

We have, for passing from a system of rectangular axes to another system having the same origin (*Coördinate Geometry*, Art. 219),

$$\left.\begin{array}{l} x = x_1 \cos(xx_1) + y_1 \cos(xy_1) + z_1 \cos(xz_1) \\ y = x_1 \cos(yx_1) + y_1 \cos(yy_1) + z_1 \cos(yz_1) \\ z = x_1 \cos(zx_1) + y_1 \cos(zy_1) + z_1 \cos(zz_1) \end{array}\right\} . \qquad (182)$$

A comparison of equations (181) and (182) gives:

$$\left.\begin{array}{l} \cos(xx_1) = \cos\varphi \cos\psi - \sin\varphi \sin\psi \cos\theta \\ \cos(xy_1) = -\sin\varphi \cos\psi - \cos\varphi \sin\psi \cos\theta \\ \cos(xz_1) = \sin\theta \sin\psi \end{array}\right\} . \qquad (183)$$

$$\left.\begin{array}{l} \cos(yx_1) = \cos\varphi \sin\psi + \sin\varphi \cos\psi \cos\theta \\ \cos(yy_1) = -\sin\varphi \sin\psi + \cos\varphi \cos\psi \cos\theta \\ \cos(yz_1) = -\sin\theta \cos\psi \end{array}\right\} . \qquad (184)$$

$$\left.\begin{array}{l} \cos(zx_1) = \sin\varphi \sin\theta \\ \cos(zy_1) = \cos\varphi \sin\theta \\ \cos(zz_1) = \cos\theta \end{array}\right\} . \qquad (185)$$

We here have nine direction cosines all expressed in terms of θ, ψ, φ. These may vary while the axes x_1, y_1, z_1, remain fixed.

The same relations may be found by the solution of spherical triangles formed by arcs of great circles joining the points where the axes pierce the surface of the sphere before referred to.

160. *Relation between the angular velocities of the body about its respective principal axes fixed in the body, and the angular velocities about the lines OZ, OZ_1, OP, respectively.*

In Fig. 129, $\frac{d\psi}{dt}$ will be the angular velocity of the body about OZ, $\frac{d\varphi}{dt}$ the angular velocity about OZ_1, and $\frac{d\theta}{dt}$, the angular velocity about OP. For the purpose of representing these quantities in the simplest manner, and for greater convenience in the following analysis, let the value of $\frac{d\psi}{dt}$, the an-

gular velocity of the system about OZ, be represented by a definite line laid off from O, positively (in this case) on the line OZ. Similarly, lay off a line of proportionate length on the line OZ_1 to represent the angular velocity about that axis; and similarly on the line OP. This representation accords with a similar representation of statical moments as shown in Article 171 of the author's *Elementary Mechanics*, and is also in accordance with the remark following equation (176) of this work. In a similar manner ω_1 will be represented by a line on x_1, ω_2 on y_1, ω_3 on z_1. This being done, the *angular velocities* may be referred to as *lines;* and the composition and resolution of angular velocities be treated in the same manner as the composition and resolution of forces, Articles 55 and 83. Hence, any one of the angular velocities will equal the sum of the projections on that axis of all the other angular velocities. Hence, by the aid of equations (185), we have:

$$\left.\begin{aligned}
\omega_1 &= \frac{d\psi}{dt}\cos ZOX_1 + \frac{d\varphi}{dt}\cos Z_1OX_1 \\
&\quad + \frac{d\theta}{dt}\cos POX_1 + \omega_2\cos Y_1OX_1 + \omega_3\cos Z_1OX_1 \\
&= \frac{d\psi}{dt}\cos(zx_1) + \frac{d\varphi}{dt}\cos 90^\circ + \frac{d\theta}{dt}\cos\varphi \\
&\qquad + \omega_2\cos 90^\circ + \omega_3\cos 90^\circ \\
&= \frac{d\psi}{dt}\sin\varphi\sin\theta + \frac{d\theta}{dt}\cos\varphi \\
\omega_2 &= \frac{d\psi}{dt}\cos ZOY_1 + \frac{d\varphi}{dt}\cos Z_1OY_1 \\
&\quad + \frac{d\theta}{dt}\cos POY_1 + \omega_1\cos X_1OY_1 + \omega_3\cos Z_1OY_1 \\
&= \frac{d\psi}{dt}\cos\varphi\sin\theta - \frac{d\theta}{dt}\sin\varphi \\
\omega_3 &= \frac{d\psi}{dt}\cos ZOZ_1 + \frac{d\varphi}{dt}\cos Z_1OZ_1 \\
&\quad + \frac{d\theta}{dt}\cos POZ_1 + \omega_1\cos X_1OZ_1 + \omega_2\cos Y_1OZ_1 \\
&= \frac{d\psi}{dt}\cos\theta + \frac{d\varphi}{dt}
\end{aligned}\right\}\ . \quad (186)$$

In a similar manner the values of $\frac{d\theta}{dt}, \frac{d\psi}{dt}, \frac{d\varphi}{dt}$, may be found, but they may be as readily found by elimination. Eliminating among (186) gives:

$$\left.\begin{aligned}\frac{d\theta}{dt} &= \omega_1 \cos\varphi - \omega_2 \sin\varphi \\ \frac{d\psi}{dt} &= \frac{\omega_1 \sin\varphi + \omega_2 \cos\varphi}{\sin\theta} \\ \frac{d\varphi}{dt} &= \omega_3 - \cot\theta\,(\omega_1 \sin\varphi + \omega_2 \cos\varphi)\end{aligned}\right\} . \qquad (187)$$

AXIS OF SPONTANEOUS ROTATION.

161. Considering the body as perfectly free, we have, according to the notation immediately following equations (165):

$$x = \bar{x} + x_1, \qquad (188)$$

from which we readily deduce:

and similarly,
$$\left.\begin{aligned}\frac{dx}{dt} &= \frac{d\bar{x}}{dt} + \frac{dx_1}{dt} \\ \frac{dy}{dt} &= \frac{d\bar{y}}{dt} + \frac{dy_1}{dt} \\ \frac{dz}{dt} &= \frac{d\bar{z}}{dt} + \frac{dz_1}{dt}\end{aligned}\right\} . \qquad (188)$$

x being a coördinate referred to a system of coördinates fixed in space, $\bar{x}$ refers to the centre of the mass in the same system of coördinates, but x_1 is a coördinate to the same particle as x, referred to a parallel system having its origin continually at the centre of the mass. Suppositions may be made arbitrarily upon any one of the three terms in equations (188), and a corresponding relation determined between the remaining

terms. Thus, if $\frac{dx_1}{dt} = 0$, and similarly for $\frac{dy_1}{dt}$ and $\frac{dz_1}{dt}$; we have

$$\frac{dx}{dt} = \frac{d\bar{x}}{dt}; \tag{189}$$

which shows that the velocity of every particle is the same as that of the centre of the mass. This is as it should be, since there will be no rotation.

If $\frac{d\bar{x}}{dt} = 0$, $\frac{d\bar{y}}{dt} = 0$, $\frac{d\bar{z}}{dt} = 0$, we have

$$\frac{dx}{dt} = \frac{dx_1}{dt}, \tag{190}$$

and similarly for the others; hence the velocity of any particle in reference to the fixed origin will be the same as that in reference to the centre of the mass. This is evidently as it should be, since the centre of the mass will, according to the hypothesis, be at rest.

Finally, if

$$\frac{dx}{dt} = 0, \quad \frac{dy}{dt} = 0, \quad \frac{dz}{dt} = 0; \tag{191}$$

we have

$$\frac{d\bar{x}}{dt} = -\frac{dx_1}{dt}, \quad \frac{d\bar{y}}{dt} = -\frac{dy_1}{dt}, \quad \frac{d\bar{z}}{dt} = -\frac{dz_1}{dt}; \tag{192}$$

Generalizing equations (191) by multiplying by l, m, n, respectively, and integrating, we have

$$lx = a, \qquad my = b, \qquad nz = c; \tag{193}$$

where a, b, c, are arbitrary constants of integration. Adding we have

$$lx + my = a + b, \quad nz + my = c + b; \tag{194}$$

which are the equations to a right line in space, and according to (191), this line will be at rest at the instant, in reference to the fixed origin. This line is called *the axis of spontaneous rotation.*

Since x_1, y_1, z_1, equations (192), refer to the same particles, it follows from these equations that the axial velocities of the particles on the axis of spontaneous rotation, in reference to the centre of the mass, are equal but directly opposite to the axial velocity of the centre of the mass in reference to the fixed origin, and hence, more briefly:

The velocity of the axis of spontaneous rotation in reference to the centre of the mass is equal but in opposite direction to that of the centre of the mass in reference to a fixed origin.

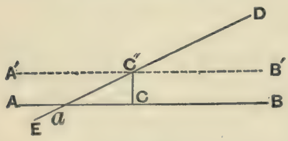

Fig. 133.

162. To illustrate this in a simple case, let AB represent a body whose centre C moves in the line CC and which rotates about its centre. If ED represent the position consecutive to AB, they will intersect in some point as a, then will a line through a perpendicular to the plane of the two positions AB and ED, be *the axis of spontaneous rotation.* Conceive the line AB to move to the consecutive parallel position $A'B'$, CC' representing the velocity of the centre; then turn the line about C' as a centre, that point a which has the same velocity backward that C had forward will be a point in the axis of spontaneous rotation. If the body be a disc moving in the plane of the paper and having a uniform rotation about its centre, the spontaneous axis will have a uniform motion parallel to the line CC'. The combined motions of translation of the entire body and of rotation about the centre of the mass may be considered as a simple instantaneous rotation about the spontaneous axis. It will also be observed that the *angular velocity* of the body about its centre will, at the instant, be the same as that of the centre about the axis of spontaneous rotation (equations (192)).

Queries. 1. Is the spontaneous axis always within the body?

2. In Fig. 133, when the bar AB falls into the line of motion CC', where will the axis of spontaneous rotation be?

3. If the centre of the line AB has a uniform velocity in a straight line, and at the same time the line has a uniform

angular velocity about its middle point, required the path of the extremity B.

4. If the line AB has a uniform velocity of 20 feet per second, and a rotary velocity of 10 turns per second, required the distance from the centre to the axis of spontaneous rotation.

5. If the uniform velocity of the centre of a disc be v feet per second, and the uniform angular velocity in the plane of the disc and of motion be ω; required the distance of the spontaneous axis from the centre of the body.

Let x be the required distance; then

$$x\omega = v;$$

$$\therefore x = \frac{v}{\omega}.$$

If r be the radius of the disc, x will be $<$, $=$, or $>$ than r, according as v is $<$, $=$, or $>$ than $r\omega$. Only one line, equations (194), fulfils the condition of a spontaneous axis.

RELATIONS BETWEEN THE SPONTANEOUS AXIS OF ROTATION, THE CENTRAL AXIS AND THE LINE OF ACTION OF THE RESULTANT.

163. Observing that x, y, z, equations (174), are coördinates in reference to the centre of a free body as an origin (the subscripts having been dropped), and hence are the same as x_1, y_1, z_1, in equations (192), we have, by substituting the values of the former in the latter,

$$\left.\begin{aligned} \omega_{z_1} y_1 &= \omega_{y_1} z_1 + \frac{d\bar{x}}{dt} \\ \omega_{x_1} z_1 &= \omega_{z_1} x_1 + \frac{d\bar{y}}{dt} \\ \omega_{y_1} x_1 &= \omega_{x_1} y_1 + \frac{d\bar{z}}{dt} \end{aligned}\right\}; \qquad (195)$$

which are the equations to the *axis of spontaneous rotation*, the origin of coördinates being at the centre of the mass. The

equations to the *axis of instantaneous rotation or central axis,* in the same notation are, equations (175), (restoring the subscripts),

$$\left.\begin{aligned} \omega_{z_1} y_1 &= \omega_{y_1} z_1 \\ \omega_{z_1} x_1 &= \omega_{x_1} z_1 \\ \omega_{y_1} x_1 &= \omega_{x_1} y_1 \end{aligned}\right\}. \qquad (196)$$

A comparison of equations (195) and (196) shows that these lines are parallel ; hence,

The spontaneous axis of rotation is parallel to the instantaneous (or central) axis.

Letting V be the velocity of the centre of the mass, a, b, c, the angles between the line of motion and the axes x, y, z, respectively, we have

$$\frac{d\bar{x}}{dt} = V \cos a, \quad \frac{d\bar{y}}{dt} = V \cos b, \quad \frac{d\bar{z}}{dt} = V \cos c. \quad (197)$$

Substituting these in (195), we find for the distance between the central axis (196) and the axis of spontaneous rotation (195), (*Coördinate Geometry*, Appendix III.), omitting numeral subscripts,

$$h_1 = \frac{V}{\omega_y} \sqrt{\frac{(\omega_x^2 + \omega_y^2)\cos^2 a + 2\,\omega_x\,\omega_z \cos a \cos c + (\omega_y^2 + \omega_z^2)\cos^2 c}{\omega_x^2 + \omega_y^2 + \omega_z^2}}\text{*}. \quad (198)$$

Let the axis of y be the axis of instantaneous rotation, then will $\omega_x = 0$, $\omega_z = 0$, and we have, dropping the subscript,

$$h_1 = \frac{V}{\omega}, \qquad (199)$$

* This may also be written

$$h_1 = \frac{V}{\omega_z} \sqrt{\frac{(\omega_x^2 + \omega_z^2)\cos^2 a + 2\omega_x\,\omega_y \cos a \cos b + (\omega_y^2 + \omega_z^2)\cos^2 b}{\omega_x^2 + \omega_y^2 + \omega_z^2}};$$

$$h_1 = \frac{V}{\omega_x} \sqrt{\frac{(\omega_x^2 + \omega_y^2)\cos^2 b + 2\omega_y\,\omega_z \cos b \cos c + (\omega_x^2 + \omega_z^2)\cos^2 c}{\omega_x^2 + \omega_y^2 + \omega_z^2}}$$

a result readily deduced from Fig. 133, since *ultimately* we would have $CC' = V = Ca \,.\, \omega$; hence $Ca = h_1 = V \div \omega$.

If the angular rotation and velocity of the centre both be uniform, h_1 (198) will be constant; and it is also evident that the linear and rotary velocities may vary proportionately in such a way as to make h_1 constant. This is readily seen in the more simple case of equation (199), and illustrated by examples 3 and 4, p. 206.

If any number of forces, F_1, F_2, F_3, etc., act upon a body producing both a translation of the centre of the mass and rotation about that centre, they will be equivalent to a single force R applied at some point of the body; for we have (Eqs. (85), (87), and (86)),

$$\left.\begin{aligned} \Sigma F \cos \alpha &= X = R \cos a \\ \Sigma F \cos \beta &= Y = R \cos b \\ \Sigma F \cos \gamma &= Z = R \cos c \end{aligned}\right\}; \qquad (200)$$

$$\therefore R = \sqrt{X^2 + Y^2 + Z^2}; \qquad (201)$$

from which R becomes known, and hence a, b, c, equations (200), also become known. To find the line of action of R; since the moments of the separate forces are known, $\Sigma F \cos \gamma \,.\, y_1 - \Sigma F \cos \beta \,.\, z_1, = L$ (say) becomes known, and similarly for the others; hence

$$\left.\begin{aligned} Zy_1 - Yz_1 &= L \\ Xz_1 - Zx_1 &= M \\ Yx_1 - Xy_1 &= N \end{aligned}\right\}; \qquad (202)$$

where x_1, y_1, z_1, are the coördinates of the line of action of R in reference to the centre of the mass as an origin; hence (202) are *the equations to the line of action of the resultant.* The third of (202) is a consequence of the other two. Since any point in the line of action of a force may be taken as its point of application, the point (x_1, y_1, z_1) may be considered as

the point of application of the resultant. To find where the resultant pierces the plane yz, make $x_1 = 0$ in (202), and we have,

$$y_1 = -\frac{N}{X}, \qquad z_1 = \frac{M}{X}; \tag{203}$$

and similarly for the other planes.

If the forces reduce to a couple, we have $R = 0$, and there will be no motion of the centre produced by this system of forces, although the centre may have a motion due to initial conditions. In this case the left members of (202) all reduce to zero, and the point of application will be found to be infinitely distant. The right members for statical equilibrium would also be zero, but if L, M, N, one or all, have finite values, they will produce rotation, and must be placed equal to the left members of (168).

The inclination, θ, between the line of the resultant (202) and the central axis (196), will be (*Coördinate Geometry*, Eq. (6), Art. 199), dropping the numerical subscripts,

$$\cos\theta = \frac{X\omega_x + Y\omega_y + Z\omega_z}{\sqrt{(X^2 + Y^2 + Z^2)(\omega_x^2 + \omega_y^2 + \omega_z^2)}}. \tag{204}$$

If $R = 0$, Equation (201), $\cos\theta = \frac{0}{0}$, which is indeterminate. If the forces are parallel to the axis of x, then $Y = 0$, $Z = 0$, $\omega_x = 0$, and

$$\cos\theta = 0;$$

that is, the action line of the resultant will be perpendicular to the central axis, and hence also perpendicular to the axis of spontaneous rotation. The shortest distance between the action line of the force, equations (202), and the spontaneous axis, equations (195) and (197), will be (Appendix III., *Coördinate Geometry*), dropping the numerical subscripts,

$$l = \frac{\left(M + \frac{Z}{\omega_z}V\cos b\right)\left(\omega_y - \frac{Y}{Z}\omega_z\right) + \left(\frac{L}{Z}\omega_z + V\cos a\right)\left(X - \frac{\omega_x}{\omega_z}Z\right)}{\sqrt{(X\omega_z - Z\omega_x)^2 + (Y\omega_z - Z\omega_y)^2 + (X\omega_y - Y\omega_z)^2}}. \tag{205}$$

As a special case, let there be a single force (or forces having a single resultant) acting parallel to the axis of x and at such a point as to produce rotation about the axis of y only, and let this be a principal axis, then we have:

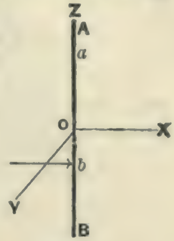

FIG. 134

$$X = R,\ Y = 0,\ Z = 0,\ L = 0,\ a = 0,\ b = 90^\circ,$$
$$c = 90^\circ,\ \omega_x = 0,\ \omega_y = \omega,\ \omega_z = 0;$$

and (205) becomes, omitting subscripts,

$$l = \frac{M}{R} + \frac{V}{\omega}. \tag{206}$$

and (204) gives

$$\cos\theta = 0; \qquad \therefore \theta = 90^\circ,$$

hence the spontaneous axis will be perpendicular to the action line of the force, will lie in the plane yz, and be at a distance from the action line equal to the value given by equation (206).

The first three conditions immediately preceding equation (206), in equations (167) give, after integrating once, R being considered constant,

$$\left.\begin{aligned} m\frac{dx}{dt} &= mV = Rt \\ \frac{dy}{dt} &= c_1 \\ \frac{dz}{dt} &= c_2 \end{aligned}\right\}; \tag{206a}$$

where m, the mass, is used to distinguish it from M, the moment, in (206) above. The velocities c_1 and c_2 are constant, and are unaffected by the force or impulse R. If c_1 and c_2 are not zero, V will not be the actual velocity of the centre, but will be the velocity produced by the resultant R. From the first of (206a) and (199) we have

$$R = \frac{mV}{t} = \frac{mh_1\omega}{t}. \tag{207}$$

In equations (180) we will have

$$\omega_1 = 0, \quad \omega_3 = 0, \quad B = \Sigma m\,(x^2 + z^2) = mk_1^2;$$

and, considering the moment of the forces as constant, we have by integrating

$$\frac{mk_1^2\omega}{t} = M; \qquad (208)$$

and hence (206) becomes

$$l = \frac{k_1^2}{h_1} + h_1 = \frac{k_1^2 + h_1^2}{h_1} \qquad (209)$$

In Fig. 134, b being the point of action of the force in reference to the body AB, and a the projection of the axis of spontaneous rotation, then $l = ab$, $h_1 = Oa$. Let $Ob = h_2$, then from (209) we have

$$l - h_1 = h_2 = \frac{k_1^2}{h_1};$$

$$\therefore\ h_1 h_2 = k_1^2; \qquad (210)$$

and since k_1 is constant, it follows that in the plane containing the action line of the force and the centre of the mass, *the spontaneous axis and point of application of the force are convertible.* In other words, if a be the point where the spontaneous axis pierces the principal plane xz when the force is applied at b in the same plane, then if the force be applied at a the spontaneous axis will pass through b. (See also Problem 8, page 209).

Although, in establishing equation (206), we have assumed a constant force applied at some point in one of the principal planes of the body, yet these conditions are not always *practicable.* Examples 4 and 5, page 206, illustrate the case when the forces are constant and parallel. Even in these cases it is necessary also to assume that the body has no initial velocity, or if it has it must be the same as might have

been produced by the action of the force, or forces, under the conditions above imposed.

But all these equations are applicable to the case of instantaneous forces, or, in other words, of impact, the initial velocities either being zero or abstracted from the above conditions.

The equation to the line passing through the origin of coördinates—which is at the centre of the mass—and the point where the action line pierces the plane yz, equations (203), will be found by dividing one equation of (203) by the other, and is—restoring the subscripts—

$$y_1 = -\frac{N_1}{M_1}z_1. \tag{211}$$

Let θ' be the angle between this line and the axis of spontaneous rotation, and we have (195), (211), (See Appendix III. of *Coördinate Geometry*),

$$\cos\theta' = \frac{\omega_{z_1}M_1 - \omega_{y_1}N_1}{\sqrt{\omega_{x_1}^2 + \omega_{y_1}^2 + \omega_{z_1}^2}\,.\,\sqrt{M_1^2 + N_1^2}}. \tag{212}$$

For the case of *impact* this may be reduced to a form which will lead to an interesting general result. Since this condition excludes accelerating forces, $X = 0$, etc., in (167) and (168), and integrating once, we have

$$M\frac{d\bar{x}}{dt} = C_1,\ M\frac{d\bar{y}}{dt} = C_2,\ M\frac{d\bar{z}}{dt} = C_3; \tag{213}$$

$$\left.\begin{aligned} \Sigma\left(my_1\frac{dz_1}{dt} - mz_1\frac{dy_1}{dt}\right) &= L_1 \\ \Sigma\left(mz_1\frac{dx_1}{dt} - mx_1\frac{dz_1}{dt}\right) &= M_1 \\ \Sigma\left(mx_1\frac{dy}{dt} - my_1\frac{dx_1}{dt}\right) &= N_1 \end{aligned}\right\}; \tag{214}$$

where L_1, M_1, N_1 are the moments of the momentum, as shown by the left members of the equations. Substituting in (214),

for $\frac{dx_1}{dt}$ etc., their values from (174) (since (174) are equally true for the origin at the centre of the mass), and making the axes principal ones by the conditions of equations (179), we finally have

$$\left.\begin{aligned} \omega_{x_1} &= \frac{L_1}{\Sigma m\,(y_1^2 + z_1^2)} = \frac{L_1}{A} \\ \omega_{y_1} &= \frac{M_1}{\Sigma m\,(z_1^2 + x_1^2)} = \frac{M_1}{B} \\ \omega_{z_1} &= \frac{N_1}{\Sigma m\,(x_1^2 + y_1^2)} = \frac{N_1}{C} \end{aligned}\right\} . \tag{215}$$

Substituting in (212) the values of ω_{x_1}, ω_{z_1}, ω_{y_1}, from (215) gives

$$\cos\theta' = \frac{\frac{B}{C} - 1}{\sqrt{\frac{M_1^2}{N_1^2} + 1} \cdot \sqrt{\left(\frac{B}{A}\cdot\frac{L_1}{M_1}\right)^2 + \left(\frac{B}{C}\cdot\frac{N_1}{M_1}\right)^2 + 1}}. \tag{216}$$

If, in (216), $B = C$, $\cos\theta' = 0$; that is if two of the principal moments of inertia equal each other, the spontaneous axis of rotation will be perpendicular to the line joining the centre of the mass and the point where the action line pierces the plane of those moment axes.

If the line of impact be parallel to the axis of x, L_1, equations (215), will be zero, which gives $\omega_{x_1} = 0$, and since $\frac{d\bar{y}}{dt}$, and $\frac{d\bar{z}}{dt}$ will also be zero, these in (195) show that the axis of spontaneous rotation will be in the plane yz. If now $B = C$, the axis of spontaneous rotation will lie in the plane of the moment axes of B and C, and be perpendicular to the line drawn from the centre of the mass normal to and intersecting the line of the impulse where it pierces the plane of these axes. Further, if the line of the impulse be in the plane xy, making any angle with those axes, then $M_1 = 0$, and $N_1 = 0$, and the

denominator becomes infinite, and hence $\theta' = 90°$; hence the axis of spontaneous rotation will be perpendicular to the plane xy, or parallel to the axis of z; and since this result is independent of the angle which the line of the impulse makes with the axis of x, it will be true when it is parallel to that axis, and in the plane xy, in which case the spontaneous axis will lie in the plane yz and still be parallel to the axis of z.

This investigation beginning at equation (211) may be still further generalized by drawing from the centre of the mass a line normal to and intersecting the line of the impulse and finding the angle θ'' between this line and the axis of spontaneous rotation, but the results so found will be but little, if any, more general than those given above.

CENTRE OF PERCUSSION.

164. It appears from Article 162 and Fig. 133, that the point a in the axis of spontaneous rotation may be considered at rest at an instant, and hence if the elements on that axis were held rigidly in space when the body is struck at b, Fig. 134, the axis would suffer no shock. Such an axis generally exists, as shown by equation (205), or more simply by equations (206), (209) and (210). Any point in the line of impact is called *the centre of percussion* in reference to the axis of spontaneous rotation, and the centre of percussion in reference to any axis is any point which may receive a blow without imparting a shock to that axis. Equation (210) enables one to find the centre of percussion in reference to any assumed axis.

In the case of a compound pendulum acted upon by gravity only, the resultant force passes through the centre of gravity of the oscillating body, and the centre of percussion is farther from the axis of suspension than is the centre of gravity. In the ballistic pendulum, the blow should be at the centre of percussion, if possible, to avoid shock upon the axis of suspension.

QUERIES. 1. Can the centre of percussion be at the centre of gravity of a body?

2. Can the axis of spontaneous rotation ever pass through the centre of gravity of the body?

CONSERVATION OF THE MOTION OF THE CENTRE OF THE MASS.

165. Any condition that will render the second members of (167) zero, gives

$$\frac{d^2\bar{x}}{dt^2} = 0, \qquad \frac{d^2\bar{y}}{dt^2} = 0, \qquad \frac{d^2\bar{z}}{dt^2} = 0\,; \tag{217}$$

which integrated gives

$$\frac{d\bar{x}}{dt} = C_1, \qquad \frac{d\bar{y}}{dt} = C_2, \qquad \frac{d\bar{z}}{dt} = C_3\,; \tag{218}$$

and

$$\left.\begin{aligned} \bar{x} &= C_1 t + B_1 \\ \bar{y} &= C_2 t + B_2 \\ \bar{z} &= C_3 t + B_3 \end{aligned}\right\}; \tag{219}$$

when C_1, etc., B_1, etc., are constants of integration. Eliminating t gives

$$\frac{x - B_1}{C_1} = \frac{y - B_2}{C_2}, \qquad \frac{x - B_1}{C_1} = \frac{z - B_3}{C_3}\,; \tag{220}$$

which are the equations to a right line.

Equations (218) give the velocities along the axes, and are constant. If C be the resultant velocity, we have

$$C = \sqrt{C_1^{\,2} + C_2^{\,2} + C_3^{\,2}}. \tag{221}$$

The second members of (167) will be zero when no forces are acting upon the system; also when a system of forces all act towards the centre but have zero for a resultant; also when the forces are the mutual actions between the parts of the system, for the resultant of such forces is zero; hence

When a body or system of bodies is acted upon by any system of forces having zero for a resultant, the motion (if any) of the centre of the mass will be rectilinear and uniform.

CONSERVATION OF THE MOMENT OF THE MOMENTUM.

166. Any condition which renders the second members of (168) equal to zero, will give, omitting the subscripts,

$$\Sigma\left(mx\frac{d^2y}{dt^2} - my\frac{d^2x}{dt^2}\right) = 0,$$

and similarly for the other two equations.

Integrating gives the three equations (214). Let the actual path of any particle be projected in the coördinate planes, and consider that projection which is on the plane xy. Let r be the radius vector to any point of the projected path, θ the angle between x and r, then, (Fig. 111),

$$x = r\cos\theta, \qquad y = r\sin\theta:$$

$$\therefore dx = -r\sin\theta d\theta + \cos\theta dr,$$

$$dy = r\cos\theta d\theta + \sin\theta dr;$$

which substituted in the third of (214) gives

$$\Sigma mr^2\frac{d\theta}{dt} = N_1. \qquad (222).$$

But $\frac{d\theta}{dt} = \omega$ is the angular velocity of the particle m about the axis of z, $r\omega$ its velocity measured in a circular arc, $mr\omega$ its circular momentum, and $r.mr\omega$ the moment of the momentum, hence $\Sigma mr^2\omega$ is the entire moment of the circular momentum. But this is the same as the moment of the momentum in the direction of motion. For if O be the origin, $OA = r$, $AOC = d\theta$, Ap, a tangent, Op perpendicular to the tangent from the origin, then will the actual moment of the momentum be

$$m\frac{ds}{dt}\,.\,Op,$$

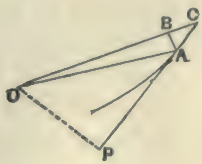

FIG. 135.

where $ds = AC$. But

$$AC : AB :: OA : Op;$$

$$\therefore AC = ds = \frac{rd\theta}{Op}r;$$

which substituted in the preceding expression gives

$$mr^2 \frac{d\theta}{dt},$$

and taking the sum of all the particles gives the left member of (222). But the right member, N_1, is constant, being the constant of integration. The same result may be shown for each of the other coördinate planes; hence the actual moment of the momentum about the central axis will be constant and equal to

$$K = \sqrt{L_1^2 + M_1^2 + N_1^2}. \qquad (223)$$

Hence, when a body or system of bodies is acted upon by forces directed towards the centre of the mass; or by mutual actions between the particles; or any system of forces in which their resultant moment is zero, the moment of the momentum is constant, and the moment of the momentum projected on ANY *plane is also constant.*

Article 150 admits of the same generalization.

THE INVARIABLE PLANE.

167. Consider that the moment of the momentum on the respective coördinate planes is represented by a line perpendicular to the plane and of proportionate length; then will K, since it is constant, equation (223), be normal to a *fixed* plane. If the axis of z be the axis of this plane, L_1 and M_1 will be zero, and $N_1 = K$; that is, the moment of the momentum on the plane xy will be a maximum.

Hence, under the same general conditions as in the preceding article, there is a fixed plane in reference to which the moment of the momentum is a maximum, and constant. The plane on which the projections of the moments of the momentum are a maximum is called the *invariable plane.*

In the solar system, knowing the positions and motions of the planets at any time, the position of the invariable plane may be found.

MUTUAL ACTION BETWEEN PARTICLES.

168. The mutual action between particles, or bodies, is of the nature of attraction, or of repulsion, or of both these forces. We will first consider attraction. Newton's law of universal gravitation is—

Two particles attract each other with a stress directly proportional to the product of their masses, and inversely as the square of the distance between them.

Thus if m and m' are two masses considered concentrated at mere points—in other words, the masses respectively of two particles—Π the stress due to the mutual attraction of two units of mass, the one upon the other, at distance unity between them, F the stress between the masses m and m' due to their mutual attractions when the distance between them is x, then

$$F = \Pi \frac{mm'}{x^2}, \tag{224}$$

where F, m, and m' are in the same units as Π. Taking the origin of coördinates on the line joining the particles, x' the abscissa of m', x'' of m, and x the distance between them at time t, then for m' we have

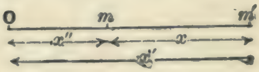

FIG. 136.

$$m' \frac{d^2x'}{dt^2} = F \cos 180° = -\Pi \frac{mm'}{x^2},$$

or

$$\frac{d^2x'}{dt^2} = -\Pi \frac{m}{x^2}; \tag{225}$$

and for m,

$$m \frac{d^2x''}{dt^2} = F \cos 0° = \Pi \frac{mm'}{x^2},$$

or

$$\frac{d^2x''}{dt^2} = \Pi \frac{m'}{x^2}, \tag{226}$$

also

$$x' - x'' = x,$$

differentiating which, gives

$$d^2x' - d^2x'' = d^2x,$$

in which substituting (225) and (226), we have

$$\frac{d^2x}{dt^2} = -\Pi\,(m + m')\,\frac{1}{x^2}. \tag{227}$$

Integrating,

$$v^2 = \frac{dx^2}{dt^2} = 2\Pi\,(m + m')\left(\frac{a - x}{ax}\right); \tag{228}$$

where a is the distance between the particles when their velocity is zero. Integrating again,

$$t = \left(\frac{a}{2\Pi\,(m + m')}\right)^{\frac{1}{2}}\left[(ax - x^2)^{\frac{1}{2}} + a\cos^{-1}\left(\frac{x}{a}\right)^{\frac{1}{2}}\right]. \tag{229}$$

In the preceding investigation the mass has been conceived to be concentrated in a mere point; it will now be extended to that of a mass of finite dimensions.

169. *To find the attraction of a homogeneous sphere upon a particle exterior to it.*

Let ABD be a spherical shell, centre C, P the position of an external particle. Conceive two consecutive radial lines drawn from P, cutting the shell in the points A and B, and join P and C.

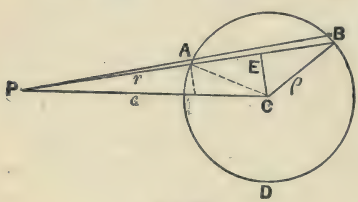

Fig 137.

Let $\theta = APC$, y = perpendicular from A on PC, ds = an element of length of the circle at A, $PA = r$, $PC = c$, $CA = CB = \rho$, $d\rho$ = thickness of the shell, δ = density, a = the radius of the sphere, and m the mass of the particle at P.

The revolution of the semicircle about PC as an axis will generate the surface of the shell. The area of a section of the shell at A whose length is ds, will be

$$d\rho\,.\,ds.$$

Let this area be revolved about PC as an axis through an angle $d\varphi$; it will generate a solid whose altitude is $yd\varphi$, and the volume so generated will be

$$yd\varphi \,.\, d\rho \,.\, ds,$$

and its mass will be

$$\delta y d\varphi\, d\rho ds.$$

The attractive stress between this element and the element at P will, equation (224), be proportional to

$$\frac{m \,.\, \delta y d\varphi d\rho ds}{r^2};$$

the component of which along the axis PC will be

$$m\delta \frac{d\rho ds y \cos\theta \,.\, d\varphi}{r^2},$$

and the attraction between the entire sphere and the particle will be

$$m\delta \int_0^a \int_0^{2\pi\rho} \int_0^{2\pi} \frac{d\rho ds y \cos\theta d\varphi}{r^2}$$

$$= 2\pi m\delta \int_0^a \int_0^{2\pi\rho} \frac{d\rho ds y \cos\theta}{r^2} \tag{230}$$

In order to integrate this, y, θ and r must be found in terms of ds; or, including ds, all must be given in terms of a single variable.

Drop the perpendicular $CE = p$, Fig. 137, from the centre, C, upon PB, then

$$p = c \sin\theta;$$

differentiating

$$dp = c \cos\theta \, d\theta; \tag{231}$$

also from the triangle PCA,

$$r^2 - 2rc \cos\theta + c^2 = \rho^2;$$

and since ρ will be constant for any particular shell, we have for that shell, by differentiating,

$$\frac{dr}{d\theta} = -\frac{rc\sin\theta}{r - c\cos\theta}.$$

From the Theory of Curves,

$$ds^2 = dr^2 + r^2 d\theta^2,$$

which, combined with the two preceding equations, gives

$$\frac{ds}{d\theta} = \frac{\rho r}{r - c\cos\theta},$$

and this with (231) gives

$$c\cos\theta\, ds = \frac{\rho r dp}{r - c\cos\theta}.$$

From the figure we have

$$r - c\cos\theta = \sqrt{\rho^2 - p^2},$$

also

$$\frac{y}{r} = \frac{p}{c},$$

or

$$y = r\frac{p}{c}.$$

These in (230) give

$$\frac{4\pi m\delta}{c^2}\int_0^a \int_0^\rho \frac{\rho d\rho p dp}{\sqrt{\rho^2 - p^2}},$$

where the former result has been multiplied by 2, since the same perpendicular CE corresponds to two elements A and B. Integrating gives

$$\frac{4\pi m\delta}{c^2}\int_0^a \rho^2 d\rho,$$

and integrating again,

$$\frac{4\pi m\delta a^3}{3c^2} = m \times \textit{volume of the sphere} \times \frac{\delta}{c^2}$$

$$= m \times \frac{\textit{mass of the sphere}}{c^2}. \qquad (232)$$

That is, the attraction between any homogeneous sphere and a particle exterior thereto varies directly as the mass of the sphere, and inversely as the square of the distance between the particle and the centre of the sphere. *It is independent of the volume of the sphere, and is the same as if the entire mass be considered as at the centre of the sphere.*

For the same reason, if the mass m be also a homogeneous sphere, it may be considered as concentrated at its centre of gravity; hence the attraction between any two spheres varies as the product of their masses conjointly and inversely as the square of the distance between their centres. This result is the same as that given by Newton in his *Principia.**

169a. Mass and stress are often referred to by the common name *pounds;* but when necessary they are distinguished as *pounds of mass*, or *pounds of force.* If two homogeneous spheres of equal size and known masses be placed at a known distance d, from each other, and under such circumstances that they are free to approach each other under their mutual attractions; then if the equal opposite forces F, applied to each sphere just sufficient to prevent their approach, be accurately measured in pounds by an accurate balance, we have from the preceding article and equation (224)

$$\Pi \frac{m^2}{d^2} = F,$$

$$\therefore \Pi = F \frac{d^2}{m^2}. \qquad (233)$$

But this method of finding Π is impracticable, chiefly on account of the difficulty of measuring the exceedingly small

* *Principia.* B. 1, Prop. LXXVI. Cor. 3.

value of Π which would result from manufactured spheres of manageable size. The usual method of finding Π and one sufficiently accurate in practice, is to consider the earth as a homogeneous sphere whose radius is the mean radius of the earth. The stress due to the attraction between the earth and any body at its surface equals the weight of the body, or $m' \times g = F$, which in equation (224) gives

$$\Pi = \frac{R^2}{m} g, \tag{234}$$

where R is the radius of the earth at the place where the body is weighed. The same result follows from equation (227) by neglecting the mass m' of the small body when compared with that of the earth, and substituting g for the acceleration, and R for x, the distance.

Let R = 20,850,000 feet, the mean radius of the earth;
E = the mass of the earth, being about $5\frac{1}{2}$ times that of an equal volume of water, page 227;
g = 32.16, the mean value of the acceleration of a free body on the earth, p. 144 of Elementary Mechanics;

then from (234) we have

$$\Pi = \frac{R^2}{E} g = \frac{R^2}{\frac{4}{3}\pi R^3 \times 5\frac{1}{2}} g \tag{235}$$

$$= \frac{3 \times 32\frac{1}{6} \times 2}{4 \times 11 \times 3.1416 \times 20,850,000} = \frac{67}{1,000,000,000} \text{lbs. nearly,}$$

if the density of the unit sphere be the average density of the earth and a foot radius. From this result Π may be found for any assumed material, since the attractive stress will vary directly as the mass, for the same volume. Equation (227) thus becomes completely determined. The unit of mass for the solar system is sometimes taken as that of the earth considered as $5\frac{1}{2}$ times that of an equal volume of water. Equation (227) is for motion in the line of centres of the bodies. If

their initial motions be not in this line, the bodies will describe orbits, as shown on pages 180 to 190. In the solar system the mass of the sun so far exceeds that of any one—or even all—the planets, that the latter are neglected in determining theoretically the character of their orbits. The value of Π does not enter this part of the problem. Having proved that the orbits are ellipses, their magnitude and position are found by determining three points from observation.

The problem of 'three bodies' subjected to mutual attractions and having arbitrary initial motions, has attracted the attention of the most eminent mathematicians. In the case of the solar system, the mass of the sun is so great that its position is considered as stationary, and the problem is reduced chiefly to that of perturbations. For a discussion of this problem see works on Theoretical Astronomy.

169b. *Repulsive forces* of the nature of electricity between two bodies are supposed to vary inversely as the square of the distance between them.

Elastic forces resist the displacement of particles from their normal position, and vary directly as the amount of displacement, whether it be a resistance to tension or compression. In this case it may be found that the orbit of a particle will be an ellipse.

In regard to constrained motion of a particle on a curve or a surface in space, equations (167) may be reduced to those of (143). If the body be finite and rotation be involved, equations (168) will also be necessary.

All the equations of statics are also contained in (167) and (168) by considering accelerations as zero. The resulting formulas are contained in the preceding pages.

PRINCIPLE OF LEAST ACTION.

170. Let m be the mass of a particle, v its velocity, ds its path during the time dt, then is the value of the definite integral

$$m\int_{s_1}^{s_2} v ds$$

defined as the "action" of the particle in passing from the former position s_1, to the latter s_2.

Since $v = ds \div dt$, the above expression reduces to

$$m\int_{t_1}^{t_2} v^2 dt.$$

The principal proposition following from this definition is: *When a system of bodies is acted upon by forces directed towards the common centre of their mass; or subject only to their mutual attractions; or by forces tending to fixed centres; then in moving from a given position to another the sum of the actions of all the bodies is less than if they had been constrained to follow any path different from the one actually described.*

The proposition in this form is due to Lagrange. It might, with propriety, have been called "the principle of least *vis viva*." It is not fruitful in the solution of problems; but the proposition once proved, leads to the general equations (167) and (168). This proposition shows that if a particle be constrained to move upon a surface under the action of parallel forces the path will be a geodetic curve.

GAUSS' THEOREM OF LEAST CONSTRAINT.

170a. *If a system of material particles be in motion, under the action of accelerating forces, the sum of the products of each particle and the square of the distance between its place at the end of time dt, and the place which it would have had under the action of the given forces, and in the same initial circumstances, if it were free, is a minimum.*

This theorem of Gauss was first given in *Crelle's Journal*, Vol. IV., 1829. It contains all the equations of motion; and its chief interest consists in presenting the subject in this peculiar form.

CHAPTER XII.

MECHANICS OF FLUIDS.

171. Matter, in regard to its physical properties, is infinitely diversified. The physical condition is conceived to depend upon the relation between the attractive and repulsive forces existing between the particles of the body. Thus, if the attractive forces exceed the repulsive, the body is a *solid*, as iron, stone, etc.; if these forces are equal, the substance is a *liquid*, as water, alcohol, etc.; and if the repulsive forces exceed the attractive, the substance is called *gaseous*, as air, hydrogen, etc. Solids are not all equally rigid. Thus, steel is vastly more rigid than jelly. As the repulsive forces increase in relation to the attractive ones, the bodies become more and more plastic, as glass, iron, lead, jelly, tar, molasses, etc., passing gradually, and it may be *imperceptibly*, from the hardest and most unyielding substance into liquids. If the particles of a liquid are not free to move among themselves, the substance is called *viscous*, as molasses, *vinegar*, etc. Liquids also pass almost *imperceptibly* into gases. There is no definite line dividing one of these classifications from the one to which it is more nearly allied, and since it is impossible to reduce the general formulas of mechanics so as to make them practically useful for all the cases which may arise, regardless of the properties of the substance considered, we make arbitrary classifications, as indicated above, and in each class treat of *ideal* substances. The ideal substances, or bodies, are perfect in themselves, and may not have a single representative in nature; still the results deduced on these hypotheses may represent, with a more or less close approximation, what actually takes place. The more nearly the real conditions approximate to the ideal conditions, the more nearly will the equations represent operations or facts in nature. Discussions involving

the *imperfect* conditions of bodies—such as viscosity, friction, elasticity, etc.—are often classed under *Applied Mechanics.*

172. Definitions.—The following are the *typical* conditions usually considered:

A rigid body, or *perfect solid,* is one in which its particles are assumed to retain their relative positions under the action of forces. The body is assumed not to change its form.

A perfect fluid is a substance in which its particles are perfectly free to move among themselves. The property of viscosity is thus excluded.

A perfect liquid is *a perfect fluid* in which the attractive and repulsive forces are equal. Water is usually taken as the type of such a substance.

A perfect gas is *a perfect fluid* in which the repulsive forces always exceed the attractive ones. Such a substance would expand indefinitely if not restrained. Air is usually taken as a *type*, though hydrogen is a more perfect gas.

A *heavy fluid* is one in which its weight is considered.

173. It was formerly supposed that water was incompressible, while it was known that air could be easily compressed, and for this reason fluids were divided into *compressible* and *incompressible*, or *elastic* and *non-elastic;* the former of which were called *gases* and the latter *liquids.* Although it has long been known that liquids are compressible, yet since the compression will be very small for pressures to which they will ordinarily be subjected, we still consider a perfect liquid as incompressible.

LAWS OF PRESSURE.

174. The pressure upon any particle of *a perfect fluid at rest* is equal in all directions, for if it were not, there would be a resultant pressure which would produce motion of the particle, since it is assumed to be perfectly free to move.

The force here considered is finite. The weight of the particle acted upon by gravity is infinitesimal, and hence if

it be at rest the upward pressure against the particle must exceed the downward by an infinitesimal amount, the amount being equal to the infinitesimal weight of the particle.

175. The pressure of *a perfect fluid* at rest, upon the surface of *the vessel containing it,* will be normal to that surface at every point of it.

For otherwise there would be a tangential component, and this would produce motion, which is contrary to the hypothesis. This proposition is also true in regard to any surface exposed to fluid pressure ; hence, if a body be immersed in a fluid, the pressure upon it will be normal at every point of its surface. The discussion in regard to the stress in a fluid might be founded on Article D, p. 154.

176. *Every external pressure upon a perfect fluid at rest will be transmitted with equal intensity to every part of the fluid in the vessel.*

For if there were any unequal pressures thus transmitted, motion would result. This is called *The Law of Equal Transmission.* It is independent of the form of the vessel. The pressures due to the *weight* of any particle of the fluid will also be transmitted equally to all parts of the fluid in the vessel *below* the point where the particle is located. It cannot be transmitted above that point, for its weight is equilibrated by the upward pressure at that point.

This proposition is illustrated by means of a vessel having closely fitting pistons at different parts of it, and the vessel filled with water, when it is found that a pressure on any one of the pistons [the pressure being p pounds per square inch] produces the same pressure per square inch on each of the other pistons. The pressure per unit area is called the *intensity* of the pressure. The proposition is not rigidly proved by this experiment, for the pistons cannot work without friction.

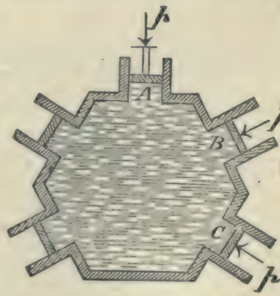

FIG. 138.

177. The pressure upon the base of a *vertical prismatic vessel* containing a perfect heavy fluid equals the weight of the fluid *plus* the pressure upon the upper surface of the fluid.

For, according to the preceding article, the pressure upon the upper base is transmitted to the base with undiminished intensity; and by the same article the entire pressure due to the weights of the particles is transmitted to the base, and must then be supported by the base.

178. The pressure upon the base of *any vessel* containing a heavy perfect fluid equals the weight of a prism of the fluid having for its base the base of the vessel, and for its altitude the height of the fluid; *plus* the pressure per unit area on the upper base into the area of the lower base. It is independent of the form of the vessel.

For, if over any element of the base a vertical prism of the fluid be conceived of the same weight as that of the liquid, and the upper surface be subjected to a pressure of the same intensity as that of the given fluid, the pressure on the element will equal the pressure on its upper base *plus* the weight of the vertical prism, Article 177. But this pressure will be transmitted equally in all directions, Article 176, and hence produces an equal pressure on every element of the base, and thus balance the real pressures. The *intensity* of the transmitted pressures is the same throughout the containing vessel, Article 176. If, therefore, the vessel be oblique, so that no real vertical prism can be erected on any element, the pressure will remain the same, depending upon the vertical height. The ideal vertical prism being suppressed the proposition will be established.

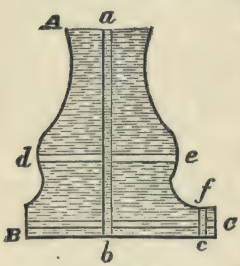

FIG 139.

If $S =$ the area of the base of the vessel,
$a =$ the depth of the fluid in the vessel,
$\delta =$ the weight of a unit of volume of the fluid,
$p =$ the pressure per unit of area upon the upper base of the fluid,
$P =$ the total pressure upon the base of the vessel;

then we have

$$P = \delta aS + pS = (\delta a + p)S. \tag{236}$$

179. Static Head.—In the preceding Article, conceive a prismatic vessel having the same base and filled with the same fluid to such a height, h, as to produce the pressure P upon the base, then

$$P = \delta hS. \tag{237}$$

which compared with equation (236) gives

$$h = a + \frac{p}{\delta}. \tag{238}$$

This value of h is called the head due to the pressure, or the reduced head. It is a height of the fluid which would produce the actual pressure upon the base. The pressure varies directly as the head. In the case of gases confined in small vessels, the weight of the fluid, compared with the external pressures, may generally be neglected, in which case we have $\delta = 0$, and (236) becomes

$$P = pS. \tag{239}$$

Gases, confined in a vessel, are always subjected to a pressure at the upper surface; but in the case of liquids, p, equation (236), may be zero, in which case a becomes the same as h in (237), and we have

$$h = \frac{P}{\delta S} = \frac{p'}{\delta}, \tag{240}$$

where p' is the pressure upon a unit of area of the base of the vessel.

180. Free Surface.—If the upper surface of a fluid is not subjected to a pressure, it is called a *free surface;* and is sometimes so called in the case of liquids when the atmosphere causes the only pressure. Liquids may have a free surface; but, according to the definition, a gas cannot have a free surface, since it may expand indefinitely. When the surface is

free the pressure on the base of the vessel is given by equation (237).

181. PRESSURE ON A SUBMERGED SURFACE.— The submerged surface may be the interior of the containing vessel, or the surface of any body within the fluid, and of any form. The normal pressure upon any element of the surface will be, Articles 176 and 179,

$$p = \delta x dA,$$

where x is the reduced head, dA the element, and δ the weight of a unit of volume of the fluid at the place of the element.

The entire normal pressure upon the surface will be

$$P = \int \delta x dA, \tag{241}$$

and if the fluid be incompressible, or if its density be uniform throughout the surface considered, δ will be constant, and we have

$$P = \delta \int x dA. \tag{242}$$

If $\bar{x}$ be the reduced head over the centre of gravity of the surface considered, then, equations (79), taking the origin in the surface of the fluid vertically over the centre of gravity of the surface, we have

$$\bar{x}A = \int x dA;$$

$$\therefore P = \delta \bar{x} A, \tag{243}$$

that is: *The normal pressure of a fluid against any surface submerged in it, equals the weight of a prism of the fluid whose base equals the area pressed and whose altitude is the reduced head over the centre of gravity of the area pressed.*

If the fluid be an incompressible liquid having a free sur-

face, the reduced head will be the *actual* head over the centre of gravity of the area pressed.

182. Resolved Pressures.—The pressure in any fixed direction will be the sum of the components of the normal pressures in that direction. If θ be the angle between the normal and required direction at any point of the surface, we have

$$\textit{resolved pressure} = \delta \int x dA \cos \theta.$$

$dA \cos \theta$ is the projection of the element on a plane normal to the required direction, which call dB; then

$$\textit{resolved pressure} = \delta \int x dB = \delta \bar{x} B, \qquad (244)$$

where x is the reduced head above the centre of gravity of the projected surface. Hence, when the projections of the elements are not superimposed

The component of pressure of a heavy perfect fluid upon any submerged surface in any direction, equals the weight of a prism of the fluid having a base equal to the projection of the surface on a plane normal to the direction, and whose altitude is the reduced head above the centre of gravity of the projected elements, each considered to be at the depth of the correponding surface elements.

183. Resultant Pressures.—If a body be submerged in a perfect, heavy fluid, the resultant of the horizontal pressures will be zero; for the projection of all the elements upon parallel planes will be equal, and hence the opposing pressures, according to the preceding article, will be equal, and their resultant zero. For the same reason the *resultant* horizontal pressures upon the interior surface of such a vessel will also be zero.

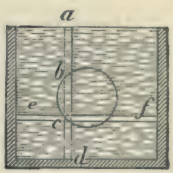

Fig. 140.

The resultant vertical pressures will also be zero, *except* that due to the weight of the displaced fluid, Article 176, hence

the resultant pressure against a submerged surface will be upward and equal to the weight of the displaced fluid.

184. Centre of Pressure.—The point through which the resultant of all the pressures upon a surface passes, is called the *centre of pressure.* Let AB be a submerged surface, CD the line of intersection of its plane with the free surface, or the upper surface of the fluid for a reduced head. Take the origin at any point on the line CD, y along ED, and x along EF. $FG =$ the head on the element $dA = dxdy$ at F. Let $\theta = FEG =$ the inclination of the surface pressed to the horizontal, then

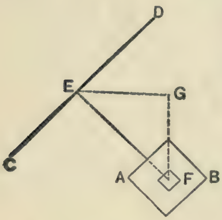

Fig. 141.

$$GF = x \sin\ \theta\,;$$

and the pressure on the element, w being the weight of a unit of volume, will be

$$wdA\ .\ x \sin\ \theta,$$

and its moment in reference to the line CD will be

$$w\,dA\,.\,x^2 \sin\ \theta,$$

and the moment for the entire surface will be

$$w \sin\ \theta \int x^2 dA.$$

Denoting the distance to the centre of gravity of the area A by $\bar{x}$, and the distance to the centre of pressure by l, we have for the reduced head over the centre of gravity:

$$\bar{x} \sin\ \theta,$$

and hence for the total pressure on the surface

$$wA\bar{x} \sin\ \theta,$$

and for the moment of the pressure,

$$wA\bar{x} \sin \theta . l;$$

hence we have, by equating the preceding values,

$$wA\bar{x}l \sin\theta = w \sin\theta \int x^2 dA;$$

$$\therefore l = \frac{\int x^2 dA}{\bar{x}A}; \tag{245}$$

that is, Articles 137 and 134, *The centre of pressure coincides with the centre of percussion, the axis of rotation being in the free surface;* or what is the same, it is the moment of inertia of the surface divided by its statical moment. It is independent of the inclination of the plane, and of the density of the fluid.

Examples.

1. *Required the entire pressure upon the interior of a cone filled with water, and standing on its base.*

Let r = the radius of the base of the cone, and

h = its altitude.

The weight of a cubic foot of water is $62\frac{1}{2}$ pounds. The area of the base will be πr^2; hence the pressure upon the base will be

$$62\tfrac{1}{2} . \pi r^2 . h.$$

The normal pressure on the concave part will be

$$62\tfrac{1}{2} . 2\pi r . \tfrac{1}{2}\sqrt{r^2 + h^2} . \tfrac{2}{3}h,$$

or

$$\tfrac{2}{3} \times 62\tfrac{1}{2} \times \pi rh \sqrt{r^2 + h^2},$$

which added to the preceding gives

$$62\tfrac{1}{2} . \pi rh \, [r + \tfrac{2}{3}\sqrt{r^2 + h^2}].$$

2. Required the normal pressure upon the interior of the cone in the preceding example when inverted.

3. Required the normal pressure upon the interior of a sphere filled with water, and compare the result with the weight of the water.

4. Find the normal pressure upon the interior of a cylindrical vessel including its base, when filled with water.

5. Find the pressure upon the interior of a cone filled with water, the axis being horizontal; the radius of the base being 1 foot and the altitude 4 feet.

6. Required the centre of pressure of a plane triangular surface immersed in a fluid, the base being in the free surface.

The moment of inertia of a triangle in reference to its base as an axis is, Article 104, Example 3,

$$\tfrac{1}{12}bd^3.$$

Its area will be $\frac{1}{2}bd$, and the distance to its centre of gravity $\frac{1}{3}d$; hence, Equation (245), we have

$$l = \frac{\frac{1}{12}bd^3}{\frac{1}{6}bd^2} = \tfrac{1}{2}d.$$

7. Required the centre of pressure of a rectangle having one end in the free surface, a being the breadth and d the depth.

8. Find the centre of pressure of a rectangle immersed vertically in a fluid, its upper end being a distance b and lower end d below the free surface, and a its breadth.

Ans. $\dfrac{2}{3}\,\dfrac{d^3-b^3}{d^2-b^2}$.

9. A cone standing on its base is filled with water; required the *vertical* pressure upon the concave part, the radius of the base being r and the altitude h.

10. In Example 9 show that the pressure upon the base minus the vertical pressure upon the concave part equals the weight of the water.

11. The concave surface of a cylinder filled with a liquid is divided by horizontal sections into n annuli in such a manner that the pressure upon each annulus equals the pressure on the base; the radius of the base being r, required the altitude and breadth of the mth annulus.

Ans. Depth $h = nr$.

Breadth of mth *annulus* $= \sqrt{rh}\,[\sqrt{m} - \sqrt{m-1}]$.

12. A rectangle, breadth 14 feet, depth 30 feet, is immersed vertically in a liquid with one end in the free surface; required the distance below the free surface of a line which divides the pressures equally.

Ans. 21.213 feet.

13. A vessel, in the form of a paraboloid of revolution, standing on its base, is filled with water; required the normal pressure on the concave part, and the vertical upward pressure on the same, the radius of the base being $1\frac{1}{2}$ feet, and altitude 4 feet.

FLOTATION.

185. Consider the case of a body in an incompressible liquid.

Let V be the volume of the body, D its density; V' the volume of the liquid displaced, and δ its density. Then, according to Article 183, the pressure vertically upward will be

$$\delta g V',$$

hence, if there be equilibrium, we have

$$gDV = g\delta V',$$

or

$$\frac{V}{V'} = \frac{\delta}{D}; \tag{246}$$

that is, *The volume of the body will be to that of the displaced liquid as the density of the liquid is to that of the body.*

If

$$\delta = D,$$

then

$$V = V';$$

or the body will be entirely submerged, but if

$$\delta > D,$$

then

$$V > V';$$

or only a part of the body will be submerged, and the body is said to float.

The intersection of the plane of the free surface with the floating body is called the *plane of flotation.*

The line joining the centre of gravity of the solid, G, and the centre of gravity C of the displaced liquid is called *the axis of flotation*, and if this line be vertical when the body is in equilibrium it is also called *the line of rest.* If the body be displaced from its line of rest, the vertical through the centre of gravity C' of the displaced liquid is called *the line of support;* and the point M where this line intersects the line of rest is called *the metacentre.*

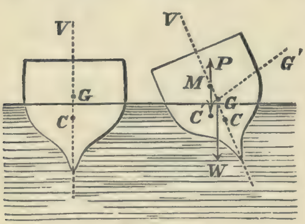

FIG. 142.

FOR THE EQUILIBRIUM of a floating body it is necessary that the line of support shall coincide with the line of rest, and the equilibrium will be *stable* if the metacentre for an indefinitely small displacement is above the centre of gravity of the solid; for in this case the reaction of the liquid along the line of support tends to turn the body toward the line of rest. If the centre of gravity of the body is below the centre of gravity of the displaced liquid, there will also be equilibrium.

186. THE DEPTH OF FLOTATION may be found by means of

equation (246) when the density and form of the body and density of the liquid are known.

For example, to find the depth of flotation of a paraboloid of revolution with the vertex downward:

Let $b =$ the radius of the base, h the altitude, and $x =$ the depth of immersion.

From the equation of the meridian section we have

$$y^2 = 2px,$$

also,

$$b^2 = 2ph;$$

$$\therefore y^2 = \frac{b^2}{h}x.$$

The volume of the solid will be

$$\tfrac{1}{2}\pi b^2 h;$$

and of the displaced liquid,

$$\tfrac{1}{2}\pi y^2 x;$$

and these substituted in Equation (246) give

$$\frac{\tfrac{1}{2}\pi b^2 h}{\tfrac{1}{2}\pi \frac{b^2}{h^2} x^2} = \frac{\delta}{D};$$

$$\therefore x = h\sqrt{\frac{Dh}{\delta}}$$

Examples.

1. Find the depth of flotation of a solid sphere whose radius is 6 inches and density $\frac{3}{4}$ that of the liquid in which it floats.

2. Find the depth of flotation of a cone whose altitude is 5 feet, radius of the base 8 inches, and whose density is one-half that of the liquid in which it floats, the axis being vertical and apex upward.

3. Required the depth of flotation of a solid paraboloid of revolution, base downward, radius of base r, altitude h, and density $\frac{2}{3}$ that of the liquid in which it floats.

4. Required the diameter of a spherical cavity in a uniform spherical shell of iron so that the depth of flotation shall be equal to the external radius of the shell, the external radius being r, and density 7 times that of the liquid in which it is submerged.

Ans. $r' = r\sqrt[3]{\frac{13}{14}}.$

5. Required the pressure necessary to just submerge a cubical block of wood each of whose edges is a feet, and whose density is $\frac{1}{2}$ that of the liquid in which it is submerged.

6. In a uniform spherical shell, external radius r, density 7, required the radius of the cavity that the plane of flotation shall be tangent to the top of the cavity.

Ans. $r' = 0.95r+.$

SPECIFIC GRAVITY.

187. If an external pressure act upon the body, either forcing it up or down, thereby producing equilibrium, we have, when the force F acts vertically down on the body,

$$F + gDV = g\delta V'; \tag{247}$$

for, the weight of the body, gDV, added to the downward pressure will equal the vertically upward pressure of the liquid. If the force F acts upward, then the upward pressure of the liquid and the force F will equal the weight of the body; hence

$$gDV = g\delta V' + F. \tag{248}$$

By means of these formulas, the weight of a body compared with the weight of an equal volume of the liquid may be determined. If the liquid used be selected as a standard, the relative weight thus found is called the *specific weight*, or *specific gravity*. The body weighed in a vacuum gives directly,

$$W = gDV; \tag{249}$$

then immersing it in the standard liquid and ascertaining the value of F necessary to produce equilibrium, we have from the preceding equations,

$$W = g\delta V' \pm F;$$

or

$$\frac{W}{g\delta V'} = 1 \pm \frac{F}{g\delta V'}, \tag{250}$$

where $+ F$ is used when the body is heavier than the liquid, and $- F$ when it is lighter.

Water at a fixed temperature (usually 60° F.) and pressure (about 29.92 in. of the barometer) is usually taken as the standard. For a further development of the subject see the Author's *Elementary Mechanics*.

188. When a mass of liquid is in motion under such conditions that its form becomes permanent, certain problems pertaining thereto may be solved by the principles of Statics. We notice the two following

PROBLEMS.

1. *A heavy perfect liquid having a free surface, is moved in a given direction with a constant acceleration, required the character of the free surface.*

Let the vessel move horizontally under the action of a constant force F, producing an acceleration f, the weight of the liquid being W—the weight of the vessel being neglected.

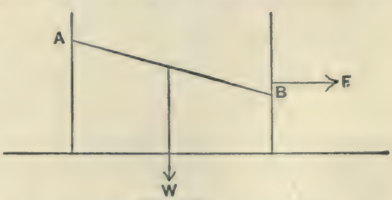

FIG. 143.

Then we have, Equations (20) and (21),

$$W = Mg, \quad \text{and} \quad F = Mf;$$

$$\therefore \frac{W}{F} = \frac{g}{f} = \text{a constant.}$$

But $F \div W$ is the tangent of the angle which the resultant of the forces of the free surface makes with the horizontal, which, being constant, shows that the slope of the free surface is constant, and hence it is a plane.

2. *A free, heavy, perfect liquid in a cylindrical vessel is rotated with a uniform velocity about its vertical axis; required the form of the free surface.*

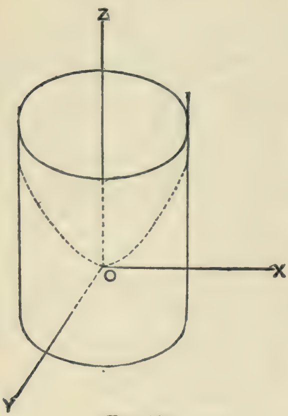

FIG. 144.

Since the forces will be the same in every meridian plane, we may consider the form in one meridian section, as xz for instance. The acceleration to which every particle is subjected is that of gravity, downwards, and the resistance to the centrifugal force radially inward.

Let ω be the constant angular velocity; then will the centrifugal force of a particle at a distance x from the axis of rotation be (Equation (142)),

$$X = -m\omega^2 x,$$

also

$$Z = -mg;$$

$$\therefore \frac{Z}{X} = \frac{g}{\omega^2 x};$$

which is the tangent of the angle of the resultant force with the horizontal:

$$\therefore \frac{dx}{dz} = \frac{g}{\omega^2 x};$$

which integrated gives,

$$x^2 = \frac{2g}{\omega^2} z,$$

the equation of the common parabola; hence the free surface is that of a paraboloid of revolution with its axis vertical.

Compare this result with that of Problem 8, page 195.

A surface to which the resultant of all the forces at each and every point of it is normal, is called a *level surface.*

QUESTIONS.—1. Will the parameters of all the paraboloids of the *level surfaces* at different depths be the same as that of the free surface?

2. Will the intensity of the pressure at the circumference of the base of the vessel be the same as at the centre?

3. If the revolution be so great as to cause the centre of the free surface to touch the base of the vessel, or even to expose a portion of the base, will the free surface still be that of a paraboloid of revolution?

REMARK.—Since the paraboloidal surface is produced by a uniform rotation, and is raised from a free horizontal surface, it will be subject to oscillations. Could steady motion be continued for a long time, these oscillations would very nearly disappear. It has been proposed by some writers to resort to this principle for the construction of very large concave mirrors for astronomical purposes, but the delicate physical conditions, and the well-nigh *perfect* mechanism necessary for its success, are obstacles in the way of this undertaking. We are not aware that such a mode of making a mirror has been attempted.

FLUID MOTION.

189. *Definitions.*—When the velocity in magnitude and direction at every point of a fluid vein is constant, the motion

is said to be "steady." In steady motion the path of any particle is called a "stream line."

If throughout a finite portion of a fluid mass the motion of any element of that portion consists of a translation and a distortion only, the motion is said to be "irrotational"—a term used by Thomson and others.

When the particles of a fluid have a "rotational" or "vortex" motion, a line drawn from point to point so that its direction is everywhere that of the instantaneous axis of rotation of the fluid, is called a "vortex-line."

If through every point of a small closed curve we draw the corresponding vortex-line, a tube will be obtained called a "vortex-tube." The fluid contained within such a tube constitutes a "vortex."

190. Bernoulli's Theorem.—Let a particle, in steady motion, trace the stream line AB; and similarly another particle at C indefinitely near A, trace the stream line CD. Trace a small closed curve through the points A and C; then will all the stream lines through the elements of the curve AC form the elements of an ideal tube, and may be replaced by an actual tube conceived to be destitute of friction; and since there is steady motion the tube will be filled at all points, and constitute an *elementary stream.* The quantity of fluid passing any two points, as A and B, in the same time, will be the same, and in this sense the flow is said to be *permanent.*

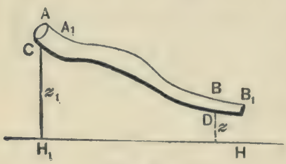

Fig. 145.

Let S_1 be the section of the tube at A, S that at B, v_1 the velocity of the flow at A, v that at B; then

$$S_1 v_1 = Sv,$$

which is the volume of the liquid flowing in a unit of time.

Let p_1 be the intensity of the pressure exerted at A, and p that at B, then will

$$p_1 S_1$$

be the entire pressure on the section S_1, and

$$p_1 S_1 v_1$$

will be the work done by p_1 in a unit of time, since the velocity may be considered constant for an element of time, and may represent the space passed over in a unit of time. Similarly, the work done by p in the opposite direction in a unit of time will be

$$-pSv.$$

Take any datum plane, above which are the ordinates z_1 to A and z to B, then the work done by gravity while the liquid is passed from the height z_1 to that of z, will be, w being the weight of a unit of volume,

$$wS_1v_1(z_1 - z).$$

The difference in the kinetic energies at A and B will be

$$\frac{wS_1v_1}{2g}(v^2 - v_1^2).$$

Since, according to the assumed condition, there is no resistance between A and B, the liquid between these points may be discarded—or, what is better, we may conceive that the liquid just before and behind the respective elements at A and B serve as a pair of perfectly flexible pistons which yield just enough to keep the tube full at all points passed by this elementary mass. Then will the entire work done upon this mass in passing from A to B equal the difference in the kinetic energies at those points, or

$$p_1S_1v_1 - pSv + wS_1v_1(z_1 - z) = \frac{wS_1v_1}{2g}(v^2 - v_1^2); \qquad (251)$$

or

$$p_1S_1v_1 + wS_1v_1z_1 + \frac{wS_1v_1}{2g}v_1^2 = pSv + wS_1v_1z + \frac{wS_1v_1}{2g}v^2. \; (252)$$

In this equation, $p_1S_1v_1$ is the potential energy of the initial pressure (Articles 26 and 151); $wS_1v_1z_1$ is the initial potential energy due to gravity in reference to any arbitrarily assumed horizontal plane; and $\frac{1}{2}\frac{wS_1v_1}{g}v_1^2$ is the initial kinetic energy of the mass $w \div g$. The sum of these will be constant for steady motion. The second member of the equation represents corresponding quantities for any point of the stream; hence

For steady motion without resistances, the sum of the potential and kinetic energies is constant.

This is *Bernoulli's* Theorem. It may be expressed in another form, for dividing Equation (252) through by wS_1v_1 we have

$$\frac{p_1}{w} + z_1 + \frac{v_1^2}{2g} = \frac{p}{w} + z + \frac{v^2}{2g}; \tag{253}$$

where $p_1 \div w$ is the head due to the initial pressure, z_1 the initial head in reference to the *datum*, $v_1^2 \div 2g$ the head due to the initial kinetic energy; and similar general expressions apply to the second member.

The sum of the heads in each member is called the *total head;* hence

For steady motion without resistances, the total head in reference to any horizontal plane is constant.

191. Discussion.—If the extremities of a stream in steady motion be in the atmosphere, the pressure at the ends, p_1 and p, will be that of the atmosphere, and in most cases will be practically equal in magnitude, but opposite in direction; for which condition (253) becomes

$$z_1 + \frac{v_1^2}{2g} = z + \frac{v^2}{2g}. \tag{254}$$

The initial velocity is often so small that it may be neglected, for which case $v_1 = 0$, and (254) becomes

$$v^2 = 2g\,(z_1 - z). \tag{255}$$

and if the datum plane passes through the lower end of the stream, we have $z = 0$, and (255) becomes

$$v = \sqrt{2gz_1}, \tag{256}$$

which is called *Torricelli's Theorem.* Comparing (255) or (256) with the first of (16), we have

In steady motion without resistances, the head due to the velocity equals the height through which a body must fall to acquire that velocity.

If the datum plane passes through the lower point considered, we have $z = 0$, and (254) gives

$$v = \sqrt{v_1^2 + 2gz_1}. \tag{257}$$

192. To represent Equation (253) graphically, let AB be a stream having a steady motion subject to different pressures along its path, and assume AH_1 equal to z_1 and draw the horizontal line H_1H. Let AC represent the head due to the pressure at A, which may be that due to the atmosphere, or the atmosphere and any other extraneous pressure, and CE the head due to the actual velocity at A. Then will AE be the total head above A, and if H_1H be the *datum*, then will H_1E be the total head above the *datum*.

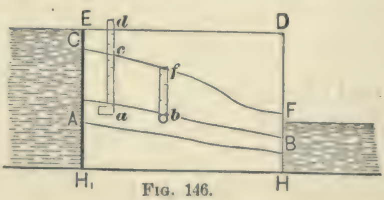

Fig. 146.

At H we will have

$$z = HB; \qquad p \div w = BF; \qquad v^2 \div 2g = DF.$$

If a vertical tube be inserted in the stream, having an opening up stream, the liquid should rise to the height ED; but if it be turned so as to be open sidewise, it would rise only to the height CF. The latter is called the *hydraulic head.*

If the initial velocity be neglected, the horizontal line ED will pass through C.

If at any point the pressure in the stream is zero, the line CF will be depressed and touch the stream at that point. Should it fall below the stream, the pressure would be negative, but as liquids have no tensile strength, this condition would destroy the "steady" motion, and the equations would not be applicable.

193. If a vessel of varying sections be left free to discharge itself, or generally if a fluid has a "steady" flow through a pipe of varying sections, the pressure of the fluid in the small sections will be less than that due to the statical head, frictional resistances being abstracted.

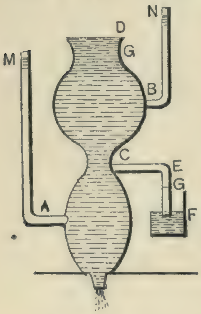

FIG. 147.

Let S_1 be the section at D, S that at B, then

$$vS = v_1S_1;$$

$$\therefore v^2 - v_1^2 = \frac{S_1^2 - S^2}{S^2}v_1^2;$$

and (253) becomes

$$\frac{p}{w} = \frac{p_1}{w} + z_1 - z + \frac{S^2 - S_1^2}{2gS^2}v_1^2; \qquad (258)$$

where $z_1 - z = BD$, $\frac{p_1}{w}$ = head due to the pressure on D, $= h_1$ (say), and $\frac{p}{w}$ the head due to the pressure at B.

If $S_1 = S$, we have

$$\frac{p}{w} = h_1 + DB; \qquad (259)$$

or the pressure will be exactly that due to the head.

If $S > S_1$, the last term of (258) will be positive, and hence

$p \div w$ will exceed the head given by (259), and we may write for this case

$$\frac{p}{w} = h_1 + BN.$$

Let the section at A be less than at D or $S < S_1$, then

$$AM = z_1 - z + \frac{S^2 - S_1^2}{2gS_1^2} v_1^2;$$

hence AM will be less than the height of the free surface above A. Also

$$\frac{p}{w} = h_1 + AM.$$

If S is so much less than S_1 that

$$\frac{p_1}{w} + z_1 - z + \frac{S^2 - S_1^2}{2gS_1^2} v_1^2$$

is negative, then $p \div w$ will be negative, and there will be a tendency to a vacuum. Let C be such a section. Then if a bent tube CEG be inserted at C, having its outer end below a liquid, the fluid from F will rise in the pipe a height FG, so that

$$FG = -\frac{p_1}{w}. \tag{260}$$

EXAMPLES.

1. A surface elementary stream of water having a velocity of 16 ft. per sec. undergoes changes in its sectional area as it passes a vessel which are proportional to the numbers 4, 6, 4, 3, 4, 6, 4; in what way can the head remain constant? Draw a vertical contour of the stream with figured dimensions or distances.

We have, since v^2 varies inversely as the square of the section,

$$v^2 \propto \tfrac{1}{16}, \tfrac{1}{36}, \tfrac{1}{16}, \tfrac{1}{9}, \tfrac{1}{16}, \tfrac{1}{36}, \tfrac{1}{16}.$$

Since the stream is in the surface p will be constant, being the pressure of the atmosphere, therefore z or the height must vary, and since the head remains absolutely the same the increase of z must be the same as the decrease of $\frac{v^2}{2g}$. $v = 16$ on entrance, and $\frac{v^2}{2g} = \frac{16 \times 16}{32 \times 2}, = 4$ ft. Hence, as the successive values of $\frac{v^2}{2g}$ are

$$\frac{v^2}{2g} = 4, \tfrac{16}{9}, 4, \tfrac{64}{9}, 4, \tfrac{16}{9}, 4,$$

$$z = 0, \tfrac{20}{9}, 0, -\tfrac{28}{9}, 0, \tfrac{20}{9}, 0;$$

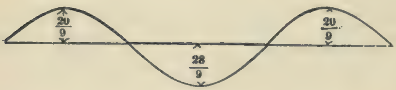

FIG. 148.

or the vertical contour will be of wave form.

2. Water flows without loss of head through a horizontal pipe of a diameter varying uniformly from 3 in. to 1 in. at smallest section, and then gradually enlarges. The velocity on entrance being 7 feet per second, what will be the minimum pressure at entrance, in order that the pipe may run full and what may be the maximum diameter of exit into the atmosphere?

Since the pipe is horizontal, $z = z_1$ in Equation (253), and according to the conditions of the example, $p = 0$ at the smallest section. We also have

$$v \text{ at entrance} = 7 \text{ feet};$$

$$\therefore v \text{ at smallest section} = 7 \times 9 = 63;$$

$$\therefore \frac{p_1}{w} + \frac{49}{2g} = \frac{(63)^2}{2g};$$

$$\therefore p_1 = 26.44 \text{ lbs.}$$

For maximum diameter of exit into the air we have in the same equation $p_1 = 0$, $p = 14.7$ lbs. per square inch, $v_1 = 63$, and $v = 63 \div d^2$;

$$\therefore \frac{14.7 \times 144}{62.5} + \left(\frac{7 \times 9}{d^2}\right)^2 \times \frac{1}{2g} = \frac{(63)^2}{2g} = 62;$$

$$\therefore d = 1.22 \text{ inches.}$$

3. Water flows from a tank through a pipe, the lower end of which is 15 feet below the entrance, the sectional area of the pipe at the tank end being twice that of the lower end. Near the tank the pipe is perforated or broken; find the head of water in the tank necessary to prevent air leaking in, or water out, through the fracture.

The pipe must be full, and the motion "steady." At the ends $p = p_1 = 34$ feet (or 14.7 lbs).

If v be the velocity at the joint, then $2v$ will be that at exit, and we will have

$$34 + \frac{v^2}{2g} + 15 = \frac{4v^2}{2g} + 34;$$

$$\therefore \frac{v^2}{2g} = 5 \text{ ft.} = \text{head in vessel.}$$

4. Water is flowing from a reservoir through a siphon pipe, the discharge end of which is 20 feet below the level of the reservoir. The diameter of the pipe is 2 inches at the discharge end, and $2\frac{1}{2}$ inches at the highest point of the siphon. Neglecting all resistances, find the height to which the siphon may be raised above the reservoir.

At A, $p_1 = 0$; at B, $p = 34$ feet; and $z_1 = z + 20 + h$, in (253).

Equating heads at A and B,

$$\frac{v^2}{2g} \times \left(\frac{16}{25}\right)^2 + 20 + h = \frac{v^2}{2g} + 34,$$

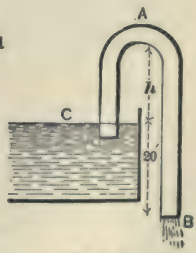

Fig. 149.

in which v is the velocity of exit.

$$\therefore h = 14 + \frac{v^2}{2g}\left\{1 - \left(\frac{16}{25}\right)^2\right\}.$$

Equating heads at C and B,

$$\frac{v^2}{2g} + 34 = 20 + 34;$$

$$\therefore v^2 = 40g,$$

and

$$h = 14 + 20\left\{1 - \left(\frac{16}{25}\right)^2\right\} = 25.7 \text{ ft.}$$

If $CB = h_1$ we would find

$$h = 34 - h_1 + \frac{v^2}{2g}\left[1 - \left(\frac{16}{25}\right)^2\right];$$

and

$$v^2 = 2gh_1;$$

from which h_1 may be found in terms of h.

FINITE RESERVOIRS AND FINITE ORIFICES.

194. If the liquid in a vessel of finite size, Fig. 150, is free to run out through an orifice in the base—or side of the vessel, the course which the elements will take may be observed by introducing into the liquid some coloring matter. In this way it is found that the fillets starting from the upper surface form curved paths which approach the orifice as a common point. In deflecting the paths of the particles, centrifugal forces will be developed, from which it follows that the opposite sides of the fluid streams will be subjected to unequal pressures, which, however, in the case of perfect fluids, will not affect the flow, except as it changes the length of the path. It is found, also, that the acceleration of the particles is different in the different elementary streams.

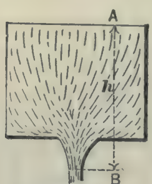

FIG. 150.

As the streams approach each other at the orifice, they interfere, and by their mutual actions produce a contraction of the vein as it leaves the orifice; the point of greatest contraction being at a distance from the orifice equal to about one-half of its diameter. This view of the problem of flow leads to an extremely difficult, if not strictly impossible, analytical solution. We therefore adopt a more simple, and at the same time a sufficiently practical hypothesis, called the principle of the *parallelism of sections*, which implies that sections parallel before motion remain so during flow, and that equal volumes pass the parallel sections in equal times.

If h be the height of the free surface of a liquid above the section of greatest contraction, we have from Equation (255)

$$v = \sqrt{2gh}; \tag{261}$$

or, *The velocity of discharge equals that of a body falling in a vacuum from the free surface to the orifice.*

Experiments show that in some cases this result is nearly realized in practice, while in some extreme cases, depending upon the form and conditions of the orifice, it is twice too large. In practice, the several cases are classified as mere orifices, short tubes, reëntrant tubes, etc., etc., and the velocity as determined by direct experiment in each of the cases, divided by the theoretical velocity, is called the *coefficient* (or *modulus*) of velocity.

Thus it is shown that when the discharge is through a thin plate, or past a well-defined sharp edge, the stream is at first contracted, forming the so-called *vena contracta.*

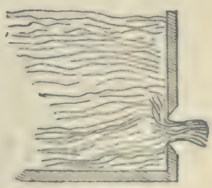

Fig. 151.

The diameter of the section of greatest contraction is about 0.8 that of the orifice; hence its section will be about 0.64 that of the orifice. For this case, the actual velocity at the section of greatest contraction is found to be about 0.97 of the theoretical, and hence we would have

$$v = 0.97\sqrt{2gh} \tag{262}$$

If the flow be through a short tube whose section is the same as that of the orifice, it is found that the quantity discharged is about 0.82 that of the theoretical, and as the section of the stream is the same as that of the tube the entire loss is due to a loss of velocity; hence, for this case

$$v = 0.82\sqrt{2gh}. \tag{263}$$

This reduction of velocity is caused by the interference of the fluid veins within the tube near and about the section where the greatest contraction would take place. If a small hole be made in the pipe, at a distance from the inside of the vessel equal to the radius of the pipe, it will be found that air will rush in, thus showing that there is a negative pressure in the pipe at that point.

Different coefficients are found for other cases.

If the velocity be the same at all parts of an orifice whose section is k, we have for the quantity of flow in a second—

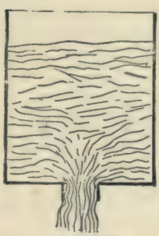

Fig. 152.

Through an orifice in a thin plate:

$$q = 0.64 \times 0.97k\sqrt{2gh} = 0.62k\sqrt{2gh}; \tag{264}$$

and through a short tube:

$$q = 0.82k\sqrt{2gh}. \tag{265}$$

195. If the orifice is so large as to cause a perceptible velocity, v_1, of the free surface, we have

$$v_1 s_1 = vs,$$

where s is the area of the contracted section and s_1 that of the free surface. This value of v_1 substituted in (254) gives:

$$v = \sqrt{\frac{2gh}{1 - \frac{s}{s_1}}}. \tag{265a}$$

If s be so small compared with s_1 that it may be neglected, we reproduce (261). If $s = s_1$, $v = \infty$, which shows that this condition cannot be realized. The liquid would drop like a free body, and hence its velocity would not be dependent upon liquid pressure.

QUESTIONS.—1. If two conical vessels of the same dimensions and filled with the same liquid discharge themselves through equal orifices, one in the base and the other at the apex, will the velocity of discharge be the same when the heads are the same?

2. In the preceding question, will the vessels empty themselves in the same time?

196. If the orifice be in the side of the vessel, and of finite dimensions, the heads of the several elementary streams or fillets will be different.

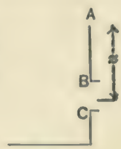

FIG. 153.

Let z be the head above any point of the orifice, then for an element having this head, we have

$$v = b\sqrt{2gz};$$

where b is the coefficient of velocity; and for the quantity flowing through this element in one second,

$$q = c\sqrt{2gz} \, . \, dzdx,$$

where c is the coefficient of discharge; and for the quantity flowing through the entire orifice, we have

$$Q = c\sqrt{2g}\iint \sqrt{z}\, dzdx, \tag{266}$$

integrated between such limits as to include the entire area of the orifice. The free surface is here supposed to remain at a constant height.

EXAMPLES.

1. *In equation* (266) *let the orifice be a rectangle, b the breadth, d the depth, $AB = h_1$, $AC = h_2$; required the quantity which would flow through the orifice in a unit of time.*

We have

$$Q = c\sqrt{2g}\int_{h_1}^{h_2}\int_0^b \sqrt{z}\,dz dx = \sqrt{2g}b\;c\int_{h_1}^{h_2}\sqrt{z}\,dz,$$

$$= \frac{2}{3}c\sqrt{2g}\,b\,(h_2^{\frac{3}{2}} - h_1^{\frac{3}{2}}), \qquad (266a)$$

2. If the upper surface of the rectangular orifice be at the free surface, the opening is called a notch; required the quantity discharged through the notch.

3. Determine the quantity which will flow from a triangular aperture, the apex being in the free surface, b being the base, h the altitude, and the base horizontal.

Ans. $\frac{2}{5}cbh\sqrt{2gh}.$

4. In the preceding example, determine the flow if the base be in the free surface.

Ans. $\frac{4}{15}cbh\sqrt{2gh}.$

5. Determine the quantity of flow, if the orifice be a circle whose radius is r, and whose centre is at a distance $h > r$ below the surface.

Ans. $\pi r^2\sqrt{2gh}\left[1 - \frac{1}{32}\left(\frac{r}{h}\right)^2 - \frac{5}{1,024}\left(\frac{r}{h}\right)^4 + \text{etc.}\right].$

6. *To determine the time in which a vessel will empty itself of a perfect liquid through an orifice in its base.*

Take the origin at the orifice, z vertical, a the area of the orifice, K the area of the free surface; then

$$Kdz$$

will be the elementary volume discharged in an element of time,

$$acvdt = ac\sqrt{2gz}\,.\,dt = -Kdz,$$

the quantity passing through the orifice in an element of time, and is negative, since t and z are inverse functions, hence

$$t = \frac{1}{ac\sqrt{2g}}\int_0^h \frac{-Kdz}{\sqrt{z}}. \qquad (266b)$$

If the section K be variable, its value must be found in terms of z before integrating.

7. To find the time in which a prismatic vessel filled with a perfect fluid will discharge itself through a mere orifice, a, in its base.

$$Ans. \frac{2Kh}{ca\sqrt{2gh}}.$$

8. A vessel, formed by the revolution of the semi-cubical parabola, $by^2 = z^3$, about its axis z, which is vertical, is filled with a liquid to the height b; to find the time in which the liquid will be discharged through a small orifice, section a, at the vertex.

Here

$$K = \pi y^2 = \frac{\pi z^3}{b};$$

and the limits are for $t = 0$, $z = h$, and for $t = t_1$, $z = 0$.

$$Ans.\ t = \frac{\pi\sqrt{2b^5}}{7ca\sqrt{g}}.$$

9. Find the time in which a paraboloid of revolution whose altitude is h and parameter p, full of liquid, will empty itself through a small orifice at its vertex, its axis being vertical.

Here $\pi y^2 = \pi px$.

$$Ans.\ t = \frac{2\pi ph^{\frac{3}{2}}}{3ac\sqrt{2g}}.$$

10. A conical vessel, the radius of whose base is r, and altitude h, is filled with a liquid; required the time in which the surface of the liquid will descend through half its altitude, the orifice being at the vertex, and the axis vertical.

Here

$$hy = rz; \quad K = \pi y^2 = \pi \frac{r^2}{h^2} z^2;$$

and the limits are $z = h$, and $z = \frac{1}{2}h$.

$$Ans.\ t = \frac{\pi r^2 h^{\frac{1}{2}} (2^{\frac{5}{2}} - 1)}{20ac\sqrt{g}}.$$

11. Find the time in which a liquid contained in a paraboloidal vessel, $y^4 = bz$, will descend equal distances h, the flow being through a small orifice whose section is a, at its vertex.

$$Ans.\ t = \frac{\pi h\sqrt{b}}{ca\sqrt{2g}}$$

The times are equal for equal heights taken anywhere along its axis. This would form a water clock in which equal times would be indicated by equal spaces.

12. A tank whose height FD is 100 feet above the level of the ground is supplied by a 1 inch pipe which communicates with a $1\frac{1}{2}$ inch horizontal pipe at the level of the ground, and is fed by a third pipe 2 inches in diameter proceeding from an accumulator 3 feet in diameter, with piston, A, 10 feet from the ground, loaded with 30 tons. Neglecting all resistances, find velocity with which the water enters the tank.

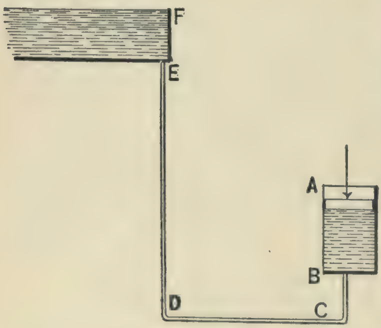

Fig. 154.

Equating heads at C and D we have

$$\frac{30 \times 2{,}240}{\frac{9 \times 3.1416}{4} \times 62.5} + 10 + 34 = 100 + \frac{v^2}{2g} + 34;$$

from which

$$v = 64 \text{ feet per second.}$$

13. Water is discharged from a vertical rectangular orifice of height d and area A. Show that approximately $Q = c\sqrt{2gh} \times A\left(1 - \frac{d^2}{96h^2}\right)$, where h is the depth of centre of orifice below the water level, the orifice being fully immersed.

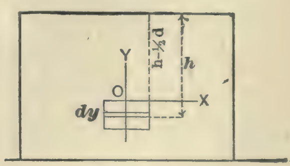

Fig. 155.

From the figure, with the origin at the upper edge of the orifice we have

$$Q = c\int_0^d \frac{A}{d}\sqrt{2g\left(h - \frac{d}{2} + y\right)}dy,$$

$$= \frac{2c}{3}\ \frac{A}{d}\ \sqrt{2g}\left[\left(h - \frac{d}{2} + y\right)^{\frac{3}{2}}\right]_0^d,$$

or

$$Q = \frac{2c}{3}\cdot\frac{A}{d}\sqrt{2g}\ \left\{\left(h + \frac{d}{2}\right)^{\frac{3}{2}} - \left(h - \frac{d}{2}\right)^{\frac{3}{2}}\right\}.$$

Developing $\left(h + \frac{d}{2}\right)^{\frac{3}{2}}$ and $\left(h - \frac{d}{2}\right)^{\frac{3}{2}}$ to four terms, and performing the operations indicated, we have

$$Q = cA\sqrt{2gh}\left(1 - \frac{d^2}{96h^2}\right).$$

14. Find the time in which the liquid in two prismatic vessels will come to the same height, one discharging itself into the other, through a short pipe connecting them at their bases. Let b be the area of each base, h their height, A the section of the pipe; and initially, let one of the vessels be filled, and the other empty.

197. If a conically convergent tube BE of the form of the *vena contracta* be attached to the orifice B, and to the small end E a tube slightly divergent be attached, it is found by experiment that the amount of flow is increased, and is even greater than if the discharge be through a simple orifice, except when the flow is into a vacuum. It appears that the liquid adheres to the sides of the tube, carrying away the particles of air from within the tube, tending to make a partial vacuum at E, or at least to diminish the internal pressure, thereby making more

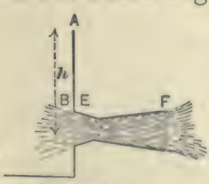

FIG. 156.

effectual the head AB and the pressure of the air on the upper surface.

The tube being entirely filled, it is the case of steady motion, and Bernoulli's Theorem applies.

If p be the pressure per unit of the atmosphere, p_1 the pressure at E, at first unknown, and considering the velocity of the surface $v_1 = 0$, equation (253), we have for the head at A (potential)

$$h + \frac{p}{w};$$

and for the head at F, z being zero,

$$\frac{v^2}{2g} + \frac{p}{w};$$

and for the head at E, z also zero,

$$\frac{v_1^2}{2g} + \frac{p_1}{w};$$

but these heads are all equal, hence

$$h + \frac{p}{w} = \frac{v^2}{2g} + \frac{p}{w} = \frac{v_1^2}{2g} + \frac{p_1}{w}; \tag{267}$$

$$\therefore v^2 = 2gh;$$

which is the velocity which it would have through a mere orifice at E; but as the section at F is larger than at E, the flow has been increased, and hence the velocity at E has also been increased. This becomes apparent from the equation

$$v_1 S_1 = vS;$$

$$\therefore v_1 = \frac{S}{S_1} v, \tag{268}$$

where v_1 and S_1 apply at E, and S at F which is larger than S_1.

We also have from (267)

$$\frac{v_1^2}{2g} = h + \frac{p - p_1}{w} = H, \tag{269}$$

or the hydraulic head exceeds h.

To find the pressure p_1, we have from (267) and (268)

$$p_1 = p + wh\left(1 - \frac{S^2}{S_1^2}\right); \tag{270}$$

and since $S_1 < S$, we have $p_1 < p$.

If $p_1 = 0$, we have

$$\left(\frac{S}{S_1}\right)^2 = \frac{p + wh}{wh}; \tag{271}$$

and if the liquid be water, $p \div w = 34$ feet, nearly, the head due to the pressure of the air, and the expression becomes

$$\left(\frac{S}{S_1}\right)^2 = \frac{34 + h}{h}.$$

Since p_1 cannot be negative for steady motion, equation (270) gives the limiting ratio of the sections, but this limit cannot be quite reached in practice.

Eytelwein found that when the mouthpiece BE was shaped like the contracted vein, followed by a divergent tube whose length was $8\frac{13}{16}$ inches long and angle of divergence $5° 9'$, that 2.5 times as much water was discharged as through a simple orifice of the size of the section at E, and 1.9 as much as through a short tube of the same section as at E.

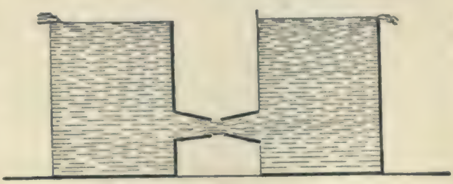

Fig. 156a.

If two vessels be connected by a tube as shown, and filled to the same height with the same liquid, and a stream be established in any manner, it will continue to flow across the space when a small portion of the tube is

removed, provided the velocity be sufficiently great, and will cease only when overcome by friction, or by the difference in heads in the vessels.

REACTION OF FLUIDS.

198. Newton's Third Law of Motion, being universal in its application, includes the action and reaction of fluids. If a heavy fluid discharges itself through an orifice in the side of a vessel suspended by a cord, the vessel will be forced away from the stream and the cord held in an inclined position. The pressure which would be exerted against the side of the vessel if the orifice be closed, is removed when the orifice is open, and the pressure continuing on the side of the vessel directly opposite the orifice forces the vessel in that direction. There is also a pressure exerted in the same direction due to the deflection of the fluid veins from their course, as will be shown hereafter. The latter is called a *reaction.*

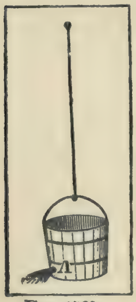

Fig. 156*b*.

199. Centrifugal Action.—The force which deflects a body from a tangent to a curve is called a *centripetal force,* and the equal opposite action of the body upon the curve is called *centrifugal force.* Strictly speaking, we should say that a force is developed between the body and the curve, which, acting one way against the body, forces it away from the curve, and in the opposite direction produces an equal pressure against the curve. If the body be attached to a central point by means of a cord, the centrifugal action would be exerted upon the fastenings at the centre; or, if there be no rigid connection, as in the case of the planets moving about the sun, the centrifugal force would be the same as if the planet moved on the concave surface of a solid coinciding in curvature with the orbit.

If the motion be in a circular arc, let m be the mass of the body, v its velocity, r the radius of the path, and φ the centrifugal force. Since the centripetal force simply changes the direction of motion, if the velocity of the body be constant,

φ will also be constant. At A the body will be moving in the direction of the tangent AB, and the centripetal force will act in the direction AO; so that the body would reach B on account of the motion in the same time that its centripetal force would draw it to D, the points B and D being consecutive to A.

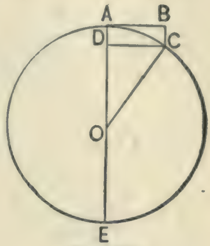

FIG. 157.

Let

$$AOC = \theta,$$

and

$$AD = x,$$

then

$$x = r\,(1 - \cos\,\theta).$$

Differentiating twice, dividing by dt^2, and multiplying by m gives

$$m\,\frac{d^2x}{dt^2} = mr\,\cos\,\theta\,\frac{d\theta^2}{dt^2};$$

the left member of which will be the value of force deflecting the body from the tangent AB, equation (21).

But at A, $\theta = 0$ and

$$\frac{d\theta}{dt} = \omega,$$

the angular velocity. Hence we have

$$\varphi = mr\omega^2$$

$$= m\frac{v^2}{r}, \qquad (272)$$

where v is the velocity along the arc of the circle.

Equation (272), is the same as equation (141), and also the last term of equation (146).

200. Resolved pressures.—If a particle be shot into a

small perfectly smooth tube having a circular bend AEB, it will exert a uniform radial pressure upon the tube; the components of which in a given direction as CX will depend upon the position of the particle. The sum of these components for a given length of arc will be the same as if that length were full of such particles all moving with the same uniform velocity; and if such a stream of particles be continuous, the pressure will be constant.

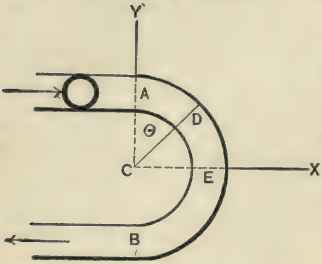

FIG. 158.

Suppose then that a steady stream of fluid passes through the tube, and let

$w =$ the weight of a unit of volume of the fluid,
$k =$ the section of the stream,
$r =$ the radius of the centre line of the tube,
$v =$ the velocity of flow,
$s = AD =$ any portion of the path from the initial point of the curve,
$\theta = ACD$,

CX parallel to the tangent at the initial point of the curve, and CY normal to it.

Friction being discarded, the velocity will be uniform throughout the tube. It is required to find the pressure in the directions CX and CY.

For any element of length, we have from the figure,

$$ds = rd\theta,$$

and from (272)

$$d\varphi = \frac{w}{g} kds \frac{v^2}{r}. \tag{273}$$

Integrating

$$\varphi = \frac{w}{g} ks \frac{v^2}{r},$$

$$= M \frac{sv}{r}, \tag{273a}$$

where M is the mass flowing into the tube in one second.

Resolving $d\varphi$, equation (273), parallel to CX and CY respectively, and integrating gives

$$X_\theta = \frac{w}{g} kv^2 \int_0^\theta \sin \theta d\theta = \frac{w}{g} kv^2 (1 - \cos \theta), \qquad (274)$$

$$Y_\theta = \frac{w}{g} kv^2 \int_0^\theta \cos \theta d\theta = \frac{w}{g} kv^2 \sin \theta. \qquad (275)$$

If the angular deviation be 90°, then $\theta = \frac{1}{2}\pi$, and (274) and (275) become

$$X_{\frac{1}{2}\pi} = \frac{w}{g} kv^2, \qquad (276)$$

$$Y_{\frac{1}{2}\pi} = \frac{w}{g} kv^2, \qquad (277)$$

which are identical. For motion through a semicircle $\theta = \pi$, and we have

$$X_\pi = 2 \frac{w}{g} kv^2, \qquad (278)$$

$$Y_\pi = 0 ; \qquad (279)$$

the last of which shows that the pressures normal to the line of the stream balance each other. For the entire circumference $\theta = 2\pi$, hence

$$\left.\begin{aligned} X_{2\pi} &= 0 \\ Y_{2\pi} &= 0 \end{aligned}\right\} . \qquad (280)$$

We observe that equations (274) to (280) are independent of the radius of the path ; hence we infer that the path may have a variable radius, but cannot be zero, since equation (272) will be infinite for $r = 0$ and v finite.

If h be the height due to the velocity v, then $v^2 = 2gh$, and (276), (277), (278), become respectively

$$X_{\frac{1}{2}\pi} = 2wkh, \qquad (281)$$

$$Y_{\frac{1}{2}\pi} = 2wkh ; \qquad (282)$$

$$X_\pi = 4wkh. \qquad (283)$$

Hence :

Deflecting a continuous stream of frictionless substance through an angle of 90° *along a curved path, produces a pressure, both in the direction of the initial motion and normal thereto, equal to a prism of the matter whose base is the section of the stream, and whose altitude is* twice *the height due to the velocity. And deflecting such a stream* 180°, *the head due to the pressure* (283) *in the direction of the initial motion will be* FOUR *times the head due to the velocity.*

If M be the mass of the liquid flowing through any section of the stream in a unit of time, then

$$\frac{w}{g} kv = M, \tag{283a}$$

and equations (274) to (278) become

$$X_\theta = Mv\,(1 - \cos\theta), \tag{284}$$

$$Y_\theta = Mv \sin\theta; \tag{285}$$

$$X_{\frac{1}{2}\pi} = Mv, \tag{286}$$

$$Y_{\frac{1}{2}\pi} = Mv; \tag{287}$$

$$X_\pi = 2Mv. \tag{288}$$

Equations (284) to (288) show that the resultant pressure due to deflecting a fluid from a rectilinear course varies *first* as the momentum of the fluid per second, and *second,* as a function of the angle through which it is deviated.

APPLICATIONS.

201. In the following applications all *resistances* due to friction, contractions, enlargements, or whorls and eddies in the stream will be discarded.

202. DISCHARGE FROM THE SIDE OF A VESSEL.—In Fig. 156*b*, considering that the fluid filaments have their origin in

the free surface, their initial direction will be vertically downward, and in order to issue from the orifice horizontally must be deflected through an angle of 90°; hence, equations (281) and (282), the pressure on the opposite side of the vessel *due to the discharge of the liquid*, will equal a prism of the liquid whose base is that of the contracted section, and whose height is twice the head above the orifice.

The correctness of this conclusion in regard to the horizontal pressure has been proved by one Peter Ewart, an English experimenter, who determined the pressure by direct measurement (*Memoirs of the Manchester Phil. Soc.*, Vol. IV.).

203. DISCHARGE VERTICALLY UPWARD.—In this case the change from the initial direction of the motion will be 180° and equations (283) and (279) are applicable, if the section of the vessel A B is sensibly the same as that of the orifice C, in which case the pressure vertically downward will equal the weight of a prism of the liquid whose base is the section of the stream, and whose altitude is FOUR times the head due to the velocity. But when the area of the orifice is small compared with that of the vessel, the velocity of the elements in A B will be small compared with those in the immediate vicinity of the orifice, and if the former be neglected, the downward reaction will be nearly that due to the deflection through a quadrant of the elements at a sensibly uniform velocity; and, hence, equal to a prism of the liquid whose base is the contracted section of the stream, and whose altitude is *twice* the head due to the velocity.

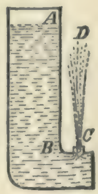

FIG. 159.

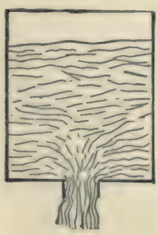
FIG. 160.

204. If the discharge be from an orifice in the base of the vessel, as in Fig. 160, in which the orifice is so small compared with the section of the vessel that the velocity of the surface may be neglected, then will the reaction be due to the deflection of the elements

in the immediate vicinity of the orifice, which may be considered as of uniform velocity and nearly all deflected through a quadrant, from a direction nearly horizontal to nearly a vertical direction; hence, the vertical reaction will be upward and nearly

$$Y = 2wkh,$$

or the upward reaction will be that due to *twice* the head producing the velocity.

205. If a stream of liquid impinge normally against a plane surface, in the immediate vicinity of the intersection of the axis of the stream and the plane an eddy or *whorl* of liquid will be formed, over which the stream will flow as along a curve and be discharged tangentially to the plane with its initial velocity. The direction of motion being changed 90°, equations (281) or (286) will determine the pressure exerted by the plane, and we have

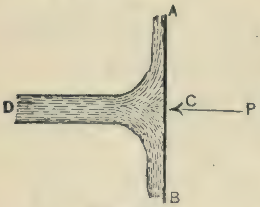

FIG. 161.

$$P = 2wkh = Mv; \tag{289}$$

that is: *The pressure exerted by a liquid stream flowing normally against a fixed plane equals (numerically) the weight of a prism of the liquid whose base is the section of the stream, and whose altitude equals twice the head due to the velocity, or equals (numerically) the momentum (per second) of the mass impinging against the surface.*

The experiments of Michelotti, Weisbach, and others show that this conclusion is very nearly realized when the impinged surface is at least six times that of the section of the stream, and placed at a distance of not less than twice the diameter of the stream from the orifice.

206. Cup vane.—If the axis of a stream coincides with that

of the axis of revolution of a surface, and impinges against the concave surface, they will be deflected as before, and flowing along the eddy as along a curve, the filaments will leave the surface tangentially, and equation (284) will be applicable.

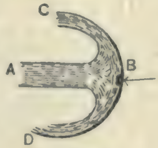

FIG. 162.

If the tangents to the surface at C and D are parallel to the axis of the stream AB, we have $\theta = \pi$, and (283) or (288) will be applicable, and we have

$$P = 4wkh = 2Mv. \tag{290}$$

The resultant pressure due to the impulse of a liquid stream against a concave hemisphere equals the weight of a prism of water whose base is the cross section of the stream, and whose height is four times the head due to the velocity.

Weisbach found by experiments with air impinging against a concave surface that the pressure was about 0.88 of the theoretical value.

207. Bent pipe.—If the tube through which the liquid flows be bent through an angle θ at the point B, an eddy, or whorl, will be formed at the angle, so that practically the flow will be along a curve, and equations (284) and (285) give the resultant pressures parallel and normal to the initial direction of the stream, which being from A toward B will be

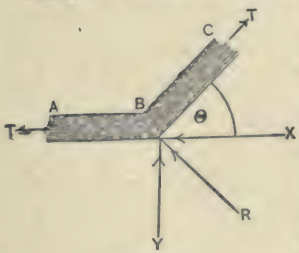

FIG. 163.

$$\left.\begin{aligned} X &= Mv(1 - \cos\theta) \\ Y &= Mv \sin\theta \end{aligned}\right\}. \tag{291}$$

The resultant R will be

$$R = \sqrt{X^2 + Y^2} = Mv\sqrt{2(1 - \cos\theta)}; \tag{292}$$

which alone will prevent the bodily movement of the tube

were no other external forces acting. The direction of R will bisect the angle ABC.

In practice the quantity of liquid discharged through a bent pipe will be less than through a straight one of the same section, on account of the contraction of the stream at the bend. The mass M will be the quantity actually discharged.

If two forces, each equal T, were acting along the branches of the tube and away from the angle, and of sufficient magnitude to produce the resultant R, we would have

$$T^2 + T^2 - 2T^2 \cos\theta = R^2 = 2M^2v^2(1 - \cos\theta),$$

$$\therefore T = Mv; \qquad (293)$$

which gives the required value of the force T.

If there be two bends, B and C, in the pipe, let two forces $T = Mv$, one at A and the other at C, act away from the angle B; they will hold the part ABC. Similarly, one at D and another at B, each equal to T and acting away from C will hold BCD; but the equal and opposite tensions along BC neutralize each other, leaving the tensions at A and D. Hence, if frictional resistances be neglected,

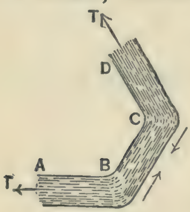

FIG. 164.

A perfectly flexible tube of uniform section having bends of any curvature, and its ends fixed in any position, will not change its curvature on account of the pressure due to the flowing of a fluid through it.

The effect of the weight of the fluid, which is not included in the above inference, would cause the tube to conform with the plane, or other surface on which it rests.

The pressure in a pipe being normal to the curve at all points will be, from Equation (o), page 139,

$$N = \frac{T}{r} = \varphi;$$

but from (272) this becomes

$$T = \frac{mv^2}{r} . r = mv^2 = Mv,$$

as before shown by (293).

208. Impinged surface inclined.—If the plane receiving the impulse be inclined an angle θ to the axis of the stream, and the stream be confined between guide plates so as to flow along the plane in the line of greatest inclination, the case will be essentially the same as that of a bent tube, and hence the pressure directly opposed to the stream, and also normal thereto, will be given by equations (284) and (285) respectively; or

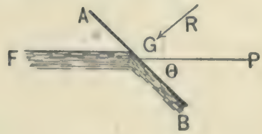

Fig. 165.

$$\left.\begin{aligned} P &= Mv\,(1 - \cos\theta) \\ Y_\theta &= Mv \sin\theta \end{aligned}\right\}. \qquad (294)$$

209. Remark.—Liquids act by impulse when suddenly changed in direction; and we have seen that the measure of this action is the same as when the stream flows along a curve of finite radius. In practice, however, certain resistances follow an impulse, due to various causes, such as the contraction of the stream, eddies, or so-called *whorls*, which make the efficiency of the fluid less than when it acts by simple pressure.

Some writers improperly use the term *impact* in this connection, as if the action were the same as that of the impact of inelastic bodies; but the impact between liquids and solids is only infinitesimal in amount, and hence eludes measurement. The true action is not an impact but a pressure—an action and a reaction of finite magnitude.

The stress between finite solids during impact is rarely sought; and, indeed cannot generally be found, for the law of action is generally unknown. To find it, the stress as a function of the time must be known, so that the value of $\int F dt$ may be found. Also the law of the distribution of the stress

throughout the section in contact must be known, from which it appears that the stress may be variable, and the finite value sought will be the sum of the infinitesimal stresses acting upon the several elements of the section of contact. Inelastic bodies have a common velocity after impact; and if a small inelastic body impinge against an indefinitely large one at rest, the motion is destroyed; but if a liquid impinge against such a body, the direction of motion is simply changed.

In the cases above considered, both the momentum and the kinetic energy of the liquid are the same after the impulse as before.

EXAMPLES.

1. The nozzle at the end of a flexible pipe of a fire engine is directed at 45° to the horizon, the pipe lying along the ground. What is the apparent increase of weight of nozzle when 150 gallons of water per minute are discharged with a velocity of 80 feet per second? Also, what is the tension of the pipe?

For the tension we have

$$T = Mv = 40 \text{ lbs.};$$

the vertical component of which will be

$$T \sin 45^\circ = 28 \text{ lbs.};$$

which is the apparent increase of weight.

2. A jet of water of sectional area A impinges beneath a horizontal plane of weight W. Find the energy of the jet required per second to support the plane?

Let m be the mass of a unit of volume of water. Then the momentum acquired per second must equal W,

$$\therefore \frac{w}{g} Av^2 = W. \qquad (1)$$

The energy of the jet is

$$wAv \times \frac{v^2}{2g}, \qquad (2)$$

but from (1),

$$v = \sqrt{\frac{W}{mA}};$$

and substituting in (2),

$$Energy = wA\sqrt{\frac{W}{mA}} \cdot \frac{W}{mA} \times \frac{1}{2g} = \frac{1}{2}\sqrt{\frac{W^3}{mA}}.$$

3. A vessel containing water and weighing 1 ton, within which the pressure is 9 atmospheres, is supported by discharging water downward; what is the diameter of the jet?

The available head, supposing the discharge against the atmosphere, is 8 atmospheres $= 8 \times 34$ feet of water.

The momentum of the jet, and the consequent reaction is

$$\frac{w}{g} Av^2 = 2{,}240;$$

or

$$\frac{\pi d^2}{4} \times \frac{62.5}{g} \times 2g \times 8 \times 34 = 2{,}240;$$

$$\therefore d = 3.474 \text{ inches.}$$

4. A jet of water strikes a fixed vane at an angle of 30° and then glances off at an angle of 60° relatively to the tangent through the initial point of impulse; required the resultant pressure on the surface, the jet delivering 600 gallons per minute, with a velocity of 10 feet per second.

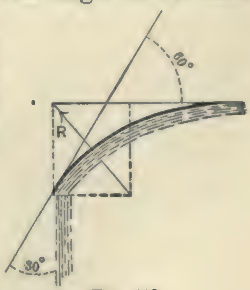

FIG. 166.

The entire deflection of the jet will be 90°; hence

$$R = \sqrt{(Mv)^2 + (Mv)^2} = Mv\sqrt{2},$$

$$= \frac{600 \times 8.4}{60 \times 32\frac{1}{6}} 10 \times \sqrt{2} = 36.9 \text{ lbs.}$$

5. A jet of water 4 inches in diameter, velocity 25 feet per second, impinges on a fixed cone at its vertex, their axes coinciding and the apex angle being 30°; find the pressure tending to move the cone.

$$P = \frac{w}{g} Qv \left(1 - \cos \frac{1}{2} \alpha\right) = 2 \frac{w}{g} Qv \sin^2 7° \, 30'.$$

$$= 3.6 \text{ lbs.}$$

6. A jet of water 4 inches wide and 1 inch thick impinges tangentially on a concave cylindrical surface of 6 inches radius, flowing over it in a stream of the same section and finally leaving it tangentially after being deflected through an angle of 60°; the velocity being 10 feet per second, what will be the intensity of the normal pressure, and the resultant force in the direction of the jet?

According to equation (273a) we have for the entire centrifugal force

$$\varphi = M \frac{sv}{r};$$

where s is the length of the arc of contact, and dividing by the area of the concave surface, which is $\frac{4}{12} s$, gives for the intensity of the pressure,

$$3M\frac{v}{r},$$

or

$$3 \times \frac{62\frac{1}{2}}{32\frac{1}{6}} \times \frac{4}{12} \times \frac{1}{12} \times 10 \times \frac{10}{\frac{1}{2}} = 32.4 \text{ pounds.}$$

The resultant pressure in the direction of the jet will be

$$Mv\,(1 - \cos 60°) = 2.7 \text{ pounds.}$$

210. If the surface impinged upon be in motion and moves in the initial direction of the stream with a uniform velocity u, the relative velocity will be $v - u$. If the same mass M as before is deflected by the vane, the pressure will be

the same as that of a stream moving with the velocity $v - u$ against a surface at rest; hence if θ be the deflection of the stream *relatively to the surface*, we have only to write $v - u$ in equations (284) and (285) to make them applicable to this case, observing that if the surface moves against the stream u will be negative. Hence, we have

$$X_\theta = M(v - u)(1 - \cos\theta) = 2M(v - u)\sin^2 \tfrac{1}{2}\theta, \quad (295)$$

$$Y_\theta = M(v - u)\sin\theta; \quad (296)$$

$$\therefore X_{\frac{1}{2}\pi} = M(v - u), \quad (297)$$

$$Y_{\frac{1}{2}\pi} = M(v - u); \quad (298)$$

and similarly for other values of θ.

WORK DONE.

211. When the body—called a vane—receiving the impulse of the stream, moves in the direction of the stream or at an acute angle therewith, work is done and energy is imparted to it and the mechanism attached thereto; but if the motion be in the opposite direction, energy will be imparted to the fluid.

If P be the pressure exerted by the fluid upon the vane whose velocity is u, the rate at which work will be done—or the *Mechanical Power* *—or simply the *Power*—will be

$$Pu, \quad (299)$$

where P will be the value given by equation (295) for this case, the entire work according to the supposition, being done in the line of x. Hence (299) and (295) give

$$Pu = X_\theta u = Mu(v - u)(1 - \cos\theta). \quad (300)$$

If the deflection be through a right angle, make $\theta = 90°$; and if two right angles, make $\theta = 180°$.

* Author's *Elementary Mechanics*, Article 99.

Equation (300) is a maximum in reference to u as a variable, when

$$u = \tfrac{1}{2}v, \tag{301}$$

which reduces (300) to

$$Pu = X_\theta \,.\, \tfrac{1}{2}v = \tfrac{1}{4}\,Mv^2\,(1 - \cos\theta), \tag{302}$$

and when the deflection is 90°, this becomes

$$Pu = X_{\frac{1}{2}\pi} \,.\, \tfrac{1}{2}v = \tfrac{1}{4}Mv^2\,; \tag{303}$$

and for 180°,

$$Pu = X_\pi \,.\, \tfrac{1}{2}v = \tfrac{1}{2}Mv^2. \tag{304}$$

Equation (304) gives an amount of work equal to the entire energy of the stream; if a fluid stream flows into a vane, like Fig. 158, or Fig. 162, where the fluid veins are completely reversed in direction, and the vane is urged forward with half the velocity of the stream, the mechanical power imparted to the vane will equal the entire energy of the stream, and the *efficiency* is said to be *perfect*. In this case the fluid leaves the vane with no actual velocity.

212. We will now deduce these results from the principle of the CONSERVATION OF ENERGY. There being no loss of energy from friction or otherwise, the kinetic energy of the stream before entering the vane will equal the energy at discharge *plus* the work imparted to the vane; or

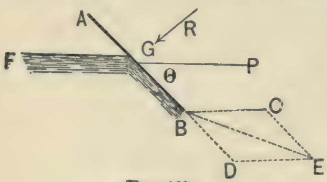

FIG. 167.

$$\tfrac{1}{2}Mv^2 = Pu + \tfrac{1}{2}MV^2, \tag{305}$$

where V is the velocity of discharge.

Let FG be the stream having a velocity v, GP the direction of motion of the vane having a velocity u in the same direction as the stream; then will $v - u$ be the velocity of the stream relatively to the vane at entrance, and since there are

no resistances, it will quit the vane tangentially with the velocity $BD = v - u$, and at the same time it will move forward with the velocity $BC = u$; hence the actual velocity will be

$$BE = V = \sqrt{(v-u)^2 + u^2 + 2u(v-u)\cos\theta}; \quad (306)$$

which in equation (305) gives

$$\tfrac{1}{2}Mv^2 = Pu + \tfrac{1}{2}M[(v-u)^2 + u^2 + 2u(v-u)\cos\theta]; \quad (307)$$

from which we find,

$$Pu = Mu(v-u)(1 - \cos\theta),$$

which is the same as equation (300). Dividing by u gives equation (295). From (305) we have

$$Pu = \tfrac{1}{2}Mv^2 - \tfrac{1}{2}MV^2, \quad (308)$$

or, *The energy imparted to the vane equals the loss of energy of the fluid.*

213. Efficiency.—*The efficiency of a stream in imparting work to a vane, is the ratio of the energy so imparted to the actual energy of the stream.*

Let e be the efficiency; then, for the preceding case, we have, equation (308),

$$e = \frac{Pu}{\tfrac{1}{2}Mv^2} = \frac{\tfrac{1}{2}Mv^2 - \tfrac{1}{2}MV^2}{\tfrac{1}{2}Mv^2} = 1 - \frac{V^2}{v^2}, \quad (309)$$

214. The Resultant Pressure in Fig. 167 will be

$$R = \sqrt{X_\theta^2 + Y_\theta^2} = M(v-u)\,2\sin\tfrac{1}{2}\theta; \quad (310)$$

and

$$\tan RGP = \frac{Y_\theta}{X_\theta} = \cot\tfrac{1}{2}\theta = \tan(90^\circ - \tfrac{1}{2}\theta);$$

$$\therefore RGP = \tfrac{1}{2}(180^\circ - \theta) = \tfrac{1}{2}FGB. \quad (311)$$

The line of the resultant pressure bisects the angle between the approaching stream FG and the plane AB.

215. General Case.—*Let the vane move in any direction.* Let AB be the vane, CD the initial direction of the stream, having a velocity $v = DF$, $u = DE =$ the velocity and direction of motion of the vane; $v_1 = BG$ = the velocity of discharge relatively to the vane, which will be tangentially to the vane; $V = BH =$ the actual velocity of discharge, and P the pressure exerted in the direction of motion DE. There being no frictional resistance, we have, as before,

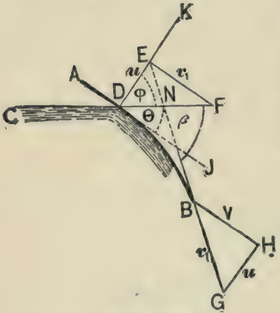

Fig. 168.

$$\tfrac{1}{2}Mv^2 = Pu + \tfrac{1}{2}MV^2. \qquad (312)$$

To find V, let $\theta = FDJ =$ the angle between the stream prolonged and a tangent to the vane at the point of impulse; $\varphi = EDF$, $\beta =$ the angle between DF and the tangent BG. The line EF joining E and F, represents the magnitude and direction of the stream relatively to the vane at D, and since there is no loss of velocity relatively to the vane while passing along it, we have

$$BG = EF = v_1 = \sqrt{v^2 + u^2 - 2vu\cos\varphi}. \qquad (313)$$

Drawing GH equal and parallel to $DE = u$, the line $BH = V$, will represent the actual velocity of discharge, and we have

$$V^2 = v_1^2 + u^2 - 2v_1u\cos BGH,$$

$$= v_1^2 + u^2 + 2v_1u\cos(\beta + \varphi); \qquad (314)$$

which after substituting v_1 from equation (313) gives

$$\left.\begin{array}{l} V^2 = v^2 - 2u(v\cos\varphi - u) \\ \qquad + 2u\cos(\beta + \varphi)\sqrt{v^2 + u^2 - 2vu\cos\varphi} \end{array}\right\}; \qquad (315)$$

hence, equations (312) and (315) give

$$Pu = Mu[v \cos \varphi - u - \cos (\beta + \varphi)\sqrt{v^2 + u^2 - 2vu \cos \varphi}]; \qquad (316)$$

$$\left.\begin{aligned} \therefore P &= M(v \cos \varphi - u - \cos (\beta + \varphi)\sqrt{v^2 + u^2 - 2vu \cos \varphi}) \\ &= M[v \cos \varphi - u - v_1 \cos (\beta + \varphi)] \end{aligned}\right\}. \qquad (317)$$

If k be the loss of energy compared with the total energy of the stream, the efficiency in imparting work to the vane will be

$$\left.\begin{aligned} e = 1 - k &= \frac{Pu}{\frac{1}{2}Mv^2} = \frac{\frac{1}{2}Mv^2 - \frac{1}{2}MV^2}{\frac{1}{2}Mv^2} = 1 - \frac{V^2}{v^2} \\ &= 2\left[\frac{u}{v}\cos\varphi - \frac{u^2}{v^2} - \frac{u}{v}\cos(\beta + \varphi)\sqrt{1 + \frac{u^2}{v^2} - 2\frac{u}{v}\cos\varphi}\right] \end{aligned}\right\}; \qquad (318)$$

from which the condition for maximum efficiency, u being variable, may be found, but the result will be too complex to be of practical value. These equations include not only all the results of the preceding cases, but also several others. They are independent of the initial slope θ of the vane, relatively to the stream, but are dependent upon the relative direction β with which it quits the vane.

216. Discussion.—Case I. Let $v \cos \varphi - u = -v_1 \cos (\beta + \varphi)$. Then equations (317), (316), (315), (318) become, respectively,

$$\left.\begin{aligned} P &= 2M(v \cos \varphi - u) \\ Pu &= 2M(v \cos \varphi - u)u \\ V^2 &= v^2 - 4(v \cos \varphi - u)u \\ e &= \frac{4(v \cos \varphi - u)u}{v^2} \end{aligned}\right\}. \qquad (319)$$

In this case $v \cos \varphi$ is the projection of the velocity v on the direction of motion of the vane u; and in all cases where work is imparted to the vane, the latter will be less than the former, and hence $v \cos \varphi - u$ will be positive, and $\cos(\beta + \varphi)$ will be negative, or $\beta + \varphi$ will exceed 90°. The second member, $v_1 \cos(\beta + \varphi)$, is the projection of v_1 on the direction of u; hence, the assumption makes the *resultant* pressure coincide with the direction of motion of the vane; also the projection equals

$$EF \cos DEF;$$

hence,

$$\cos KEF = -\cos(\varphi + \beta);$$

$$\therefore KEF = 180° - (\varphi + \beta). \qquad (320)$$

The internal angles at D and N of the triangle EDN equal the external angle KEN, or

$$KEN = \varphi + \beta;$$

$$\therefore DEN = 180° - KEN = 180° - (\varphi + \beta); \qquad (321)$$

hence (320), (321),

$$DEN = KEF,$$

as it should, since, as shown above, v_1 may be laid off on EF or EN; and if the angles be all measured from DE prolonged, we have, equation (321): *The angle between the direction of motion of the vane, and that of the stream relatively to the vane at the point of impulse, must equal the supplement of the angle between the former line and that of the direction of the stream relatively to the vane where it quits it.*

The line GB produced will not generally pass through E, but in any case the relation of the angles will be that given above, since a line may be drawn through E parallel to BG.

The speed of maximum efficiency will be found by making the second member of the last of equations (319) a maximum in reference to u as a variable, which requires that

$$u = \tfrac{1}{2} v \cos \varphi; \qquad (322)$$

which reduces (319) to

$$\left.\begin{aligned} P &= Mv\cos\varphi \\ Pu &= \tfrac{1}{2}Mv^2\cos^2\varphi \\ V &= v\sin\varphi \\ e &= \cos^2\varphi \end{aligned}\right\}; \tag{323}$$

and P, Pu, and e will be a maximum for $\varphi = 0$, when V will be zero.

CASE II. *Let the vane be flat and oblique to the stream.*

For this case β becomes θ, and by substituting the latter for β in (315), (316), 317), (318), the required results will be found. The equations will be of the same form as for the general case.

CASE III. *Let the vane be flat and move normally to the vane.*

For this case $\theta = \beta$, and

$$\varphi + \beta = 90°; \tag{324}$$

and equations (315), (316), (317), (318) become

$$\left.\begin{aligned} P &= M(v\cos\varphi - u) \\ Pu &= M(v\cos\varphi - u)u \\ V^2 &= v^2 - 2u\,(v\cos\varphi - u) \\ e &= \frac{2(v\cos\varphi - u)u}{v^2} \end{aligned}\right\}. \tag{325}$$

The speed for maximum efficiency will be when

$$u = \tfrac{1}{2}v\cos\varphi, \tag{326}$$

which reduce (325) to

$$\left.\begin{aligned} P &= \tfrac{1}{2}Mv\cos\varphi \\ Pu &= \tfrac{1}{4}Mv^2\cos^2\varphi \\ V^2 &= (1 - \tfrac{1}{2}\cos^2\varphi)v^2 \\ e &= \tfrac{1}{2}\cos^2\varphi \end{aligned}\right\} . \qquad (327)$$

Case IV. *Let the vane be flat and normal to the stream, the vane moving at an angle φ with the stream.*

In this case $\beta = 90°$, and we have from (317), (316), (315),

$$\left.\begin{aligned} P &= M\,(v\cos\varphi - u + \sin\varphi\sqrt{v^2 + u^2 - 2vu\cos\varphi}) \\ Pu &= M\,(v\cos\varphi - u + \sin\varphi\sqrt{v^2 + u^2 - 2vu\cos\varphi})u \\ V^2 &= v^2 - 2(v\cos\varphi - u)\,u \\ &\quad - 2u\sin\varphi\sqrt{v^2 + u^2 - 2vu\cos\varphi} \end{aligned}\right\} . \qquad (328)$$

Case V. *Let the vane be flat and normal to the stream, and move in the direction of the stream.*

In this case $\beta = 90°$, $\varphi = 0$, which in (315) to (318), or (325), (326), (327), give

$$\left.\begin{aligned} P &= M\,(v - u) \\ Pu &= M\,(v - u)u \\ V^2 &= v^2 - 2uv + 2u^2 \\ &= u^2 + (v - u)^2 \\ e &= \frac{2(v - u)u}{v^2} \end{aligned}\right\} ; \qquad (329)$$

and for maximum efficiency,

$$\left.\begin{aligned} u &= \tfrac{1}{2}v \\ P &= \tfrac{1}{2}Mv \\ Pu &= \tfrac{1}{4}Mv^2 \\ V^2 &= \tfrac{1}{2}v^2 \\ e &= \tfrac{1}{2} \end{aligned}\right\} . \qquad (330)$$

CASE VI. *Let the vane be a hemisphere moving in the direction of the stream.*

Then $\beta = 180°, \qquad \varphi = 0,$

and is a sub-case of *Case* I., hence equations (319) to (323) become

$$\left.\begin{aligned} P &= 2M(v-u) \\ Pu &= 2M(v-u)u \\ V^2 &= (v-2u)^2 \\ e &= \frac{4(v-u)u}{v^2} \end{aligned}\right\}; \qquad (331)$$

and for maximum efficiency

$$\left.\begin{aligned} u &= \tfrac{1}{2}v \\ P &= Mv \\ Pu &= \tfrac{1}{2}Mv^2 \\ V &= 0 \\ e &= 1 \end{aligned}\right\}. \qquad (332)$$

CASE VII. *Let the vane move normally to the stream, and* $\beta = 90°$.

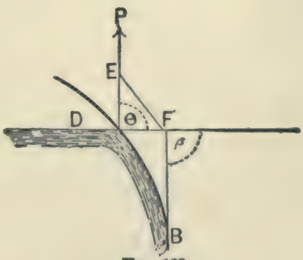

FIG. 169.

Then $\varphi = 90°,$

$$\beta + \varphi = 180°,$$

and we have

$$\left.\begin{aligned} P &= M(-u + \sqrt{v^2 + u^2}) \\ P\,u &= M(-u + \sqrt{v^2 + u^2})u \\ V^2 &= v^2 + 2(u - \sqrt{v^2 + u^2})u \\ e &= \frac{2(-u + \sqrt{v^2 + u^2})u}{v^2} \end{aligned}\right\} \qquad (333)$$

This is, substantially, a sub-case of *Case* IV., when $\varphi = 90°$.

HYDRAULIC MOTORS.

217. A hydraulic motor is any device or machine by means of which the energy of water may be utilized. These machines are of various classes, among which we notice the *water pressure engine*, in which the water passes into and out of the machine through ports, similarly to steam in the steam engine; *water wheels*, in which the water flows against floats at the outer circumference of the wheel; *turbines*, in which the water passes through the wheel either radially or parallel to the axis of the wheel; and *reaction wheels* in which the wheel is actuated by the reaction of the water passing through it.

In making an application of the preceding principles, it will be necessary only to describe the general features of the construction of each machine, leaving the details to those treatises which make a specialty of this subject. That part of the stream which impinges against the vane of a motor will be called a *jet* to distinguish it from the stream which supplies the reservoir.

218. A SINGLE VANE moving with a velocity u normally before a jet having a velocity v, will be impinged upon with the relative velocity $v - u$, and hence the quantity impinging per second will be

$$mk(v - u),$$

and the relative momentum will be

$$P = \frac{w}{g} k(v - u)^2; \qquad (334)$$

and the work done on the vane will be

$$Pu = \frac{w}{g} k(v - u)^2 u; \qquad (335)$$

which is a maximum for

$$u = \tfrac{1}{3} v. \qquad (336)$$

It will be observed that this analysis gives the same result

as substituting $\frac{w}{g}k\,(v - u)$ for M in equation (300), and making $\theta = 90°$. The analysis just given assumes that only a portion of the water passing a given fixed section of the jet is utilized in producing work, as will be the case when only a single vane receives the impulse. A single vane will not be used in practice, but instead thereof a succession of vanes at short intervals; in which case it is assumed that the entire mass of the water in the jet is directly involved in producing work.

VERTICAL WHEELS.

219. VERTICAL WHEELS are such as revolve in a vertical plane, and hence their axes of rotation are horizontal. The principal classes are *Undershot Wheels*, *Breast Wheels*, and *Overshot Wheels*.

220. THE UNDERSHOT WHEEL has vanes, floats, or buckets at its circumference for receiving the water. The water flows under the wheel nearly horizontally, and after doing its work upon the vanes escapes through the tail-race. There are two kinds—one with flat vanes, the other with curved vanes.

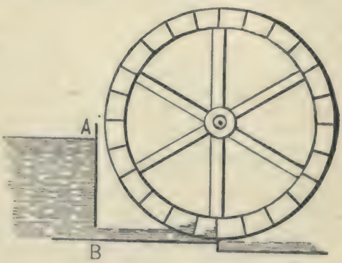

FIG. 170.

221. IN UNDERSHOT WHEELS WITH FLAT VANES it is assumed that the water impinges upon the vanes normally and leaves them tangentially; which suppositions being very nearly realized, are sufficiently accurate in practice, and make equations (329) and (330) directly applicable.

Let

$Q =$ the number of cubic feet per second of the water discharged by the jet against the vanes;

Then will

$$62\tfrac{1}{2}Q = \text{the pounds of water discharged per second};$$

and

$$M = \frac{62\frac{1}{2}Q}{32\frac{1}{6}} = 2Q \text{ approximately}, \tag{337}$$

and the mechanical power imparted to the wheel will be, 2d of Equations (329),

$$Pu = \frac{62\frac{1}{2}}{32\frac{1}{6}}\, Q\,(v - u)u\,; \tag{338}$$

which may be written :

$$Pu = 62\tfrac{1}{2}Q\left[\frac{v^2}{2g} - \frac{u^2}{2g} - \frac{(v-u)^2}{2g}\right]. \tag{338a}$$

The three terms within the parenthesis are expressions of heads due to velocities, and their algebraic sum is called the *effective head;* hence

The effective head is the head due to the velocity of entrance of the jet LESS *the head due to the velocity in the wheel at exit, and also* LESS *the head lost at the entrance of the jet into the vane, and the two latter must be a minimum that the effective head shall be a maximum.*

This principle may be generalized and used in all cases; thus: Let

H = the total available head of the fall,
h = the head due to changes of velocity at the entrance of the float;
h' = the head due to the velocity with which the water quits the wheel;
h'' = the head due to frictional resistances;
h''' = the head due to whirls or other causes.

Then

$$Pu = 62\tfrac{1}{2}Q[H - h - h' - h'' - h''']\,; \tag{338b}$$

and

$$e = \frac{Pu}{WH}, \tag{338c}$$

the horse power developed will be

$$HP = \frac{60 \times 62\frac{1}{2}}{32\frac{1}{6} \times 33,000} Q(v-u)u; \tag{339}$$

which will be *theoretically a maximum* when the circumference of the wheel has one-half the velocity of the jet; in which case (339) becomes

$$HP = \frac{62\frac{1}{2}}{32\frac{1}{6} \times 2,200} Qv^2,$$

$$\frac{Qv^2}{1,132} \text{ nearly,}$$

$$= \frac{5}{88} Qh, \tag{340}$$

where h is the head due to the velocity v. If S be the area in feet of the section of greatest contraction, we have

$$Q = vS = S\sqrt{2gh};$$

$$\therefore HP = 0.4558\, Sh^{\frac{3}{2}}. \tag{341}$$

The theoretical maximum efficiency in this case will be (5th of equations (330))

$$e = \tfrac{1}{2},$$

but this considerably exceeds the value realized in practice. The several losses due to the contraction of the vein, the clearances about the wheel through which the water escapes without acting upon the wheel, the lack of normal impulse, the imperfect action of a thick vein, and other causes, combine to reduce the theoretical efficiency. Experiments show that a

well-made wheel will realize about 60 per cent. of the theoretical efficiency, giving for fair practice, equation (339),

$$HP = 0.60 \frac{60 \times 62\frac{1}{2}}{32\frac{1}{6} \times 33,000} Q(v - u)u. \qquad (342)$$

For a practical maximum efficiency we have

$$e_1 = \text{from } 0.30 \text{ to } 0.38. \qquad (343)$$

The *power* of the wheel is independent of its size; and hence may be so proportioned as to make a desired number of revolutions per minute.

222. The Poncelet Wheel.—M. Poncelet, a celebrated French scientist, improved the undershot wheel by making the vanes so curved that the water upon leaving them would flow backward relatively to the vane. For this case, equations (331) and (332) are applicable, and we have theoretically,

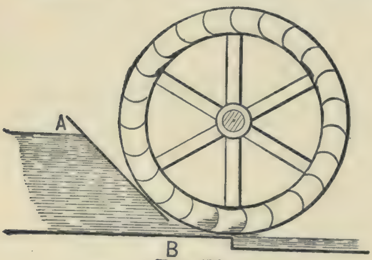

Fig. 171.

$$HP = 2 \frac{60 \times 62\frac{1}{2}}{32\frac{1}{6} \times 33,000} Q(v - u)u; \qquad (344)$$

and for a theoretical maximum

$$HP = \frac{Qv^2}{566} \text{ nearly.} \qquad (345)$$

If the velocity of the circumference of the wheel be $\frac{1}{2}$ that of the jet, and the buckets be so curved as to completely reverse the direction of motion of the water, the wheel should be perfect, or

$$e = 1, \qquad (346)$$

but these conditions not being realized, combined with the

losses noticed in the preceding Article, make the *practical* efficiency in good wheels about

$$e_1 = 0.60. \tag{347}$$

223. A BREAST WHEEL is one in which the water is admitted at some point opposite the face of the wheel, and the water retained in the wheel to the lowest point by means of a curved trough, or passage way. The jet will enter the wheel with a velocity, but if the buckets be properly curved and the velocity of the wheel be properly regulated, there will be, theoretically, no loss from this cause. After the water has entered the wheel it will act by its weight through the remaining height. Let h be the head due to the velocity, and $h_1 = CD =$ the height through which the water acts by its weight; then will the maximum theoretical power be

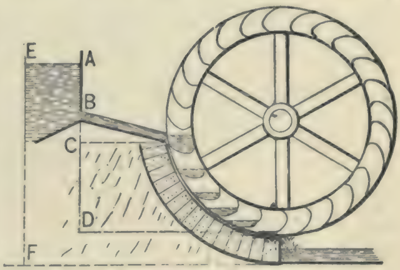

FIG. 172.

$$62\tfrac{1}{2} Q (h + h_1). \tag{348}$$

If $H = EF$ be the entire fall of the water, the theoretical power will be

$$62\tfrac{1}{2} QH,$$

and hence the *theoretical efficiency* will be

$$e = \frac{h + h_1}{H}. \tag{349}$$

Experiments with the best wheels of this class have given an *actual efficiency* of

$$e_1 = .75 \text{ to } .80. \tag{350}$$

224. OVERSHOT WHEELS receive the water at or near their highest part, and retain it in buckets during its descent. If

retained only by buckets, the water spills out before reaching the bottom of the wheel, and thus produces loss of efficiency; and if the water be retained by a curved trough, this class of wheels will not differ essentially from the breast wheel in theory.

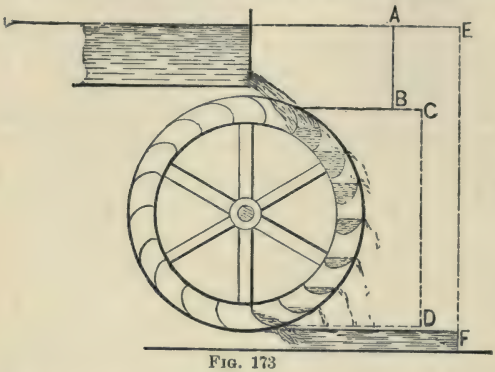

FIG. 173

If the water acts by its weight through the height $CD = h_1$ then will the mechanical power be

$$Pu = AM(v - u)u + 62\tfrac{1}{2}Qh_1, \qquad (351)$$

where A is a coefficient whose value in the case of flat vanes will be unity, as shown by the 2d of Equations (329), and for a complete reversal of direction of the jet will be 2, as shown by the 2d of equations (331); hence the real value of A will be more than 1 and less than 2.

If the velocity of the circumference, u, of the wheel be half that of the water when it enters the wheel, and there be no loss from the impulse, then $\dfrac{A(v-u)u}{2g}$ will equal $AB = h$; and we have

$$Pu = 62\tfrac{1}{2}Q(h + h_1), \qquad (352)$$

the same as (342), and the efficiency of the wheel would be perfect. But there will necessarily be some loss from impulse, and an additional amount from clearance in the curved trough, and still more from the imperfection of action due to the thickness of the jet, all of which combine to make the practical efficiency of wheels well constructed and properly operated, of

$$e_1 = 0.75 \text{ to } 0.80. \qquad (353)$$

EXAMPLES.

1. Water approaches an overshot wheel with a velocity of 12 feet per second, the buckets moving with a velocity of 6 feet per second where the fall is 20 feet, but on account of emptying the buckets 4 feet are wasted; find the efficiency, and if the supply be 600 cubic feet per minute, find the horse power.

The relative velocity of 6 feet per second in the race is equivalent to a head $h = \frac{6^2}{2g}$. The height through which the water acts by its weight will be $20 - 4 = 16$ feet.

Then, for useful work per second, we have, making $A = 1$ in equation (351),

$$mQ(12 - 6) \times 6 + wQ \times 16,$$

or

$$36 \frac{w}{g} Q + 16wQ.$$

The energy at command will be

$$\tfrac{1}{2} mQ(12)^2 + 20wQ,$$

or

$$72 \frac{wQ}{g} + 20\, wQ;$$

and we have for the efficiency,

$$\frac{36 + 16 \times 32\frac{1}{6}}{72 + 20 \times 32\frac{1}{6}} = .77.$$

The horse power will be

$$\frac{(36 + 16 \times 32\frac{1}{6}) \times 62\frac{1}{2} \times 600}{33{,}000 \times 32\frac{1}{6}} = 19.5.$$

2. The section of a stream of water falling vertically over a dam 12 feet high is one square foot at the foot of the fall; required the horse-power. *Ans.* 37.87 +.

3. If a breast wheel whose diameter equals the height of fall, receives a stream of water, section S, directly opposite its centre, if there be a loss of 5 per cent. of head on account of the clearance at the bottom of the wheel, and of 10 per cent. of velocity at the contracted section; required the horse power and the maximum efficiency.

If the vanes be flat, and the impulse normal, the velocity of the circumference being u, equations (329) will be applicable, but in properly constructed vanes, Case V., equations (331) will be applicable. Assuming the latter, let R be the radius of the wheel, then

$$v = 0.90 \sqrt{2gR},$$

and, for maximum efficiency $u = \frac{1}{2}v$, or

$$u = 0.45 \sqrt{2gR};$$

and the power of the impulse will be (3d of (332)),

$$Pu = \frac{62\frac{1}{2}Q \times (0.90)^2 \times 2gR}{2g} = 0.81 \times 62\tfrac{1}{2}QR;$$

the work done by the weight will be

$$62\tfrac{1}{2} \times Q \times 0.95R;$$

hence the total work will be

$$110QR. \qquad (a)$$

The work done by $62\frac{1}{2}Q$ falling a height $2R$ will be

$$2 \times 62\tfrac{1}{2}QR;$$

hence the maximum efficiency will be

$$e = \frac{110QR}{135QR} = 0.88; \qquad (b)$$

To find the horse power, substitute

$$Q = vS = 7.2\sqrt{R}\,S., \tag{c}$$

in equation (a), and find

$$HP = \frac{110 \times 7.2\sqrt{R}\,.\,S \times R \times 60}{33{,}000},$$

$$= 1.44SR^{\frac{3}{2}}.$$

4. Required the horse power and efficiency of an undershot wheel when the velocity of its circumference is $\frac{1}{4}$ that of the jet; the quantity of water discharged being 600 cubic feet per minute.

5. Required the horse power and efficiency of an undershot wheel when the velocity of its vanes is $\frac{3}{4}$ that of the jet, when $Q = 600$ cubic feet per minute.

6. If the discharge of water against a Poncelet wheel be 100 cubic feet per second, and the velocity of its perimeter be $\frac{1}{4}$ that of the jet, required the horse-power and efficiency.

7. Determine the horse power and efficiency in the preceding example if the velocity of its perimeter be $\frac{3}{4}$ that of the jet.

For the horse power we have, equation (344),

$$HP = 2\frac{60 \times 62\frac{1}{2}}{32\frac{1}{6} \times 33{,}000}\,.\,100\,(v - \tfrac{3}{4}v)\tfrac{3}{4}v,$$

$$= \tfrac{2}{15}v^2 \text{ nearly.}$$

For the efficiency we have

$$e = \frac{2M(v - \frac{3}{4}v)\frac{3}{4}v}{\frac{1}{2}Mv^2} = \tfrac{3}{4}.$$

8. If the velocity of the perimeter of a Poncelet wheel be **5**

feet per second, and the discharge of the jet be 200 cubic feet per minute; determine the section of the jet for maximum efficiency.

9. In the preceding example, determine the section of the jet when the efficiency is one fourth the theoretical.

10. In the Poncelet wheel, the velocity, u, of the circumference being fixed, and the required horse power, HP, being given; find the head due to the velocity when the efficiency is a maximum, and the section of the orifice, the coefficient of contraction of the vein being 0.62.

11. At what velocities of the wheel will the efficiency be zero?

12. If a wheel is being run at a velocity to produce a maximum efficiency with a jet of 25 square inches, how much must the section of the jet be increased to produce the same work when the wheel revolves with $\frac{3}{8}$ the former velocity?

225. It will be seen from the preceding discussion that if the water can be made to enter the wheel or act upon its vanes, without shock and leave it without velocity, the highest efficiency will be produced; and if in addition to these there be no losses from friction or clearance, or other imperfect conditions, the efficiency will be unity, and the wheel is said to be perfect.

Some writers distinguish between *impulse* and *reaction;* the former being applied to the direct action of the water in producing the impulse; and the latter to the effect due to the change of motion after the water has entered the vane, and hence may be affected by friction and the direction with which it leaves the vane. According to the view presented above, the action of liquids, whether of impulse or pressure, is of the nature of a reaction.

The effective head is that portion of the actual head which would produce the work done by acting upon the wheel without loss of any kind. Thus the coefficient of M in equations (300), (316), (323), (325), etc., divided by g, and $(v - u) u$

$\div g$ in equations (338) and (344) are effective heads for the respective cases here cited.

REACTION MOTORS.

226. Jet Propeller.—If water issues horizontally from the side of a vessel, it will, by virtue of its reaction, move the vessel in the direction opposite to that of the jet, when the resulting pressure is sufficient to overcome the resistances. The vessel may move with a uniform velocity, or with an acceleration either positive or negative depending upon the action of the external forces. In this analysis assume that the quantity of water in the vessel is sensibly constant during the action, and discard frictional resistances of the issuing jet.

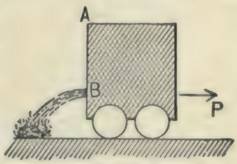

Fig. 174.

Let $h = AB =$ the static head above the orifice B,

$v =$ the velocity of discharge due to the head h,

$u =$ the velocity of the vessel,

$f =$ the acceleration of the vessel,

$h_1 =$ a head which statically would produce the same pressure at the orifice B as is produced by the acceleration,

$u_1 =$ the velocity produced by the head h_1,

$V =$ the velocity of discharge relative to the orifice;

then,

$$v^2 = 2gh, \qquad u_1^2 = 2fh_1, \tag{354}$$

and

$$V^2 = v^2 \pm u_1^2. \tag{355}$$

The actual velocity of the jet will be

$$V - u;$$

and the rate of doing work at the instant the velocity is u will be

$$Pu = M(\sqrt{v^2 \pm u_1^2} - u)u. \tag{356}$$

If the velocity of the vessel is uniform, or the acceleration is so small that it may be neglected, then $u_1 = 0$, and

$$Pu = M(v - u)u, \tag{357}$$

which will be a maximum for $u = \frac{1}{2}v$, in which case the maximum power developed will be

$$Pu = \tfrac{1}{4}Mv^2. \tag{358}$$

The efficiency will be

$$E = \frac{U}{U + \frac{1}{2}M(v - u)^2} = \frac{2u}{u + v}, \tag{359}$$

which has no maximum.

But if the vessel on the arm be connected by a pipe to a supply chamber about the axis of rotation, the pressure at the orifice will be increased on account of the centrifugal force of the water in the pipe due to the rotation. To find this pressure, and the equivalent head, assume that the inner end of the pipe is at the axis of rotation, BC, Fig. 175, and let $a = CF$, the distance of the orifice from the axis, m the mass of a prism of the water whose base is unity and length unity, and ρ a variable radius, then will the mass of an element of length be $m\,d\rho$ and the centrifugal force will be

$$\int_0^a m\,d\rho \cdot \omega^2\rho = \tfrac{1}{2}m\omega^2\rho^2.$$

If h_1 be the head which would produce this pressure, then

$$mgh_1 = \tfrac{1}{2}m\omega^2\rho^2;$$

$$\therefore \omega^2\rho^2 = 2gh_1 = u^2, \tag{360}$$

since $\omega\rho = u$;

$$\therefore V^2 = v^2 + u^2, \tag{361}$$

$$Pu = M(\sqrt{v^2 + u^2} - u)u. \tag{362}$$

227. Barker's (or Seguin's) Mill consists of a vertical hollow shaft communicating with hollow transverse arms.

The water is admitted into the upper end of the hollow shaft, and passing downward, flows horizontally into the hollow arms, escaping horizontally through orifices near their extremities, one orifice being on one side of one arm, and the other on the opposite side of the other arm. The deflection of the water from the vertical to the horizontal direction will cause both a horizontal and vertical pressure, as shown in Article 200; but the horizontal pressures will neutralize each other, while the vertical pressure will be resisted by the support, and the water will flow into the arms with a velocity unaffected by these pressures, and, if the arms are stationary, the water will be discharged at the orifices with a velocity due to the head in the vertical shaft. But when the arms are rotating, the water in them will be forced radially outward on account of the centrifugal force developed, and thus the head of discharge relatively to the orifice will be increased.

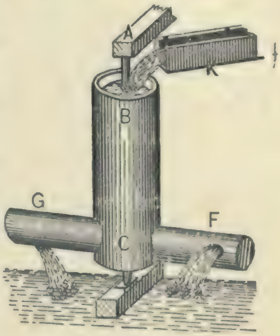

FIG. 175.

As the water approaches the orifices which are near the ends of the arms, it will be deflected through an angle of 90°, thus producing a pressure radially outward, and also transversely to the arms, the former of which will be resisted by the solid parts of the machine, and the latter will produce the rotary motion. The two reactions—one at each end of the outer extremities of the arms—produce a couple, the moment of which will equal the moment of the resistances overcome by the machine.

Let $v =$ the velocity due to the head $h = CB$,

$a = CF$,

$\omega =$ the angular velocity of the arms,

$u =$ the actual velocity of the orifices G and $F = a\omega$,

$V =$ the velocity of discharge normal to the arm relative to the orifice,

$h_1 =$ the head due to the velocity u.

Then, as shown in the preceding Article,

$$V^2 = u^2 + v^2 = a^2\omega^2 + v^2; \tag{363}$$

and the actual velocity of discharge will be

$$V - u = \sqrt{a^2\omega^2 + v^2} - a\omega; \tag{364}$$

and the pressure due to the reaction will be

$$P = M\left(\sqrt{a^2\omega^2 + v^2} - a\omega\right); \tag{365}$$

and the mechanical power resulting, will be

$$Pu = M\left(\sqrt{a^2\omega^2 + v^2} - a\omega\right)a\omega; \tag{366}$$

and the theoretical efficiency will be

$$e = \frac{M\left(\sqrt{a^2\omega^2 + v^2} - a\omega\right)a\omega}{\frac{1}{2}Mv^2}$$

$$= \frac{2a\omega}{V + a\omega}; \tag{367}$$

which has no maximum in reference to ω as a variable, but approaches unity as a limit as ω is increased *indefinitely*. In practice it is found that the efficiency is a maximum when $a\omega$ is about equal to v; and making $a\omega = v$, and neglecting all losses, except that due to a loss of actual velocity of discharge, we find for the efficiency,

$$e = 0.82 +. \tag{368}$$

If there were several orifices at distances a', a'', a''', etc., from the axis of the shaft, producing reactions P', P'', P''', etc., giving expressions similar to those in equation (365), and R the resistance overcome with a uniform velocity w at a distance b from the axis of the same shaft, then would the mo-

ments of the reactions equal the moment of the resistance overcome, and hence

$$Rb = P'a' + P''a'' +, \text{etc.},$$

$$= M'(\sqrt{a'^2\omega^2 + v^2} - a'\omega)\,a'$$

$$+ M''(\sqrt{a''^2\omega^2 + v^2} - a''\omega)\,a'' +, \text{etc.}; \quad (369)$$

and the mechanical power will be

$$Rb\omega = M'(\sqrt{a'^2\omega^2 + v^2} - a'\omega)\,a'\omega +, \text{etc.} \qquad (370)$$

An analysis of the second member of the last equation gives the following:—the quantity $\sqrt{a'^2\omega^2 + v^2} - a'\omega$ is the actual velocity of discharge as shown by equation (364); $M'(\sqrt{a'^2\omega^2 + v^2} - a'\omega)$ is the momentum of the mass M', as shown by the definition in Article 27; and $M'(\sqrt{a'^2\omega^2 + v^2} - a'\omega)\,a'$ is the moment of the momentum, as shown in Article 166, or in Article 168 of the *Elementary Mechanics.* Hence the entire expression, $M'(\sqrt{a'^2\omega^2 + v^2} - a'\omega)\,a'\omega$, is *the moment of the momentum of the mass* M' *multiplied by the angular velocity*, and similarly for the other terms. Separating the expression into other terms, we have $M'(\sqrt{a'^2\omega^2 + v^2})a'$, which is the moment of the momentum of the mass M' having the velocity $\sqrt{a'^2\omega^2 + v^2}$ in reference to the orifice; and $M'a'\omega \cdot a'$, which is the moment of the momentum of the mass M' having an actual velocity $a'\omega$, equal to the actual velocity of the orifice, which, being in a direction opposite to that of the discharge, will be negative. Each of these multiplied by the angular velocity, ω, will give the corresponding mechanical power developed; and similarly for the other terms containing M'', M''', etc.

In this case the water enters the arm without angular velocity, still the principle shown in Article 166 is general, for the effect produced is the result of the mutual action and reaction between the water and arm. Hence,

The work done by the reaction of water in passing through a motor equals the difference of its moment of momentum on entering and quitting the wheel.

This machine has been the subject of many improvements. To remove as much as possible the resistance of the water in passing through the arms, sudden turns, eddying, etc., the arms have been made large where they join the body of the shaft, and gradually contracted and curved as they pass outward toward the openings. Instead of arms, there may be a disc having openings which permit the water to escape tangentially, which form is known as Whitelaw's Turbine. In the form shown in Fig. 175, the pivot at the lower end of the shaft supports the mill, and all the water running through it, thus producing much friction. This objection has been in part removed by introducing the water into the mill from the under side, thus producing an upward pressure, and counterbalancing, in part, the weight of the machine. But notwithstanding all these improvements, the mill has been superseded by other more efficient motors.

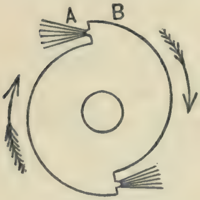

Fig. 176.

228. *To determine the pressure due to the centrifugal force.*—Let A be the uniform cross-section of the arms, ω the angular velocity, p the pressure per unit, m the mass of a unit of volume, ρ any distance from the axis of rotation, the initial and terminal values of which are r_1 and r_2 respectively, then will the mass of a transverse section be

$$mAd\rho,$$

and the centrifugal force will be

$$mAd\rho \cdot \omega^2 \rho,$$

which will equal

$$Adp;$$

$$\therefore \quad \int_{p_0}^{p} Adp = mA\omega^2 \int_{r_1}^{r_2} \rho d\rho;$$

or
$$p - p_0 = \tfrac{1}{2} m\omega^2 (r_2^2 - r_1^2), \qquad (371)$$

which is the pressure required.

If w be the weight of a unit of volume, we have

$$2g\left(\frac{p}{w} - \frac{p_0}{w}\right) = 2g\,(h - h_0) = \omega^2 (r_2^2 - r_1^2), \qquad (372)$$

where h is the head due to the velocity $r_2\omega$, and h_0 that due to $r_1\omega$. The velocity due to the difference of these heads will be

$$u = \sqrt{2g\,(h - h_0)} = \omega\sqrt{r_2^2 - r_1^2}. \qquad (373)$$

The pressure per unit, p, will be independent of the form of the longitudinal section of the arm, provided it be constantly full. Assume the axis of the arm to be in a horizontal plane, and the transverse sections variable, then when stationary Bernoulli's Theorem, equation (253), would give the pressure p at any distance r from the axis of rotation; then if rotation exist, the pressure at that point will be increased an amount produced by the centrifugal force between the distances r_1 and r. Letting p_1, v_1, r_1, be one set of contemporaneous values, and p, v, r, another set; finding $\frac{p}{w}$ from equation (253) after making $z = z_1$, and adding thereto the head due to the increased head given by (372), we have for any point of the rotating arm,

$$\frac{p}{w} = \frac{p_1}{w} + \frac{v_1^2}{2g} - \frac{v^2}{2g} + \frac{\omega^2 r^2}{2g} - \frac{\omega^2 r_1^2}{2g}. \qquad (374)$$

1. Find the horse power, number of revolutions per minute, gallons of water discharged, and efficiency of a Barker's mill, neglecting friction. Area of nozzles 1 sq. ft., radius 3 feet, total head 30 feet, speed of orifices $\frac{3}{4}$ths that due to the head.

$$u = \tfrac{3}{4}\sqrt{2g \times 30} = 32.96 \text{ feet per second};$$

$$2\pi n \times 3 = 32.96 \times 60,$$

$$\therefore\ n = 104.6;$$

$$V = \sqrt{u^2 + 2gh}, \text{ or } V = 54.8;$$

$$Gls = \frac{54.8 \times 1{,}728 \times 60}{231} = 24{,}594.$$

$$Efficiency = \frac{2u}{u + V} = 0.75.$$

$$\text{Horse power} = 140.7.$$

2. A rotating wheel receives 2 tons of water per minute at a radius of 2 feet, and delivers it at a radius of $3\frac{1}{2}$ feet; on entrance, the water is rotating with a velocity of 14 feet per second, and on delivery it is rotating in the opposite direction with a velocity of 4 feet per second. Find the couple exerted on the wheel, and if the wheel makes 200 revolutions per minute, find the HP developed.

$$\textit{Moment of momentum on entry} = \frac{2 \times 2240}{60 \times 32} \times 14 \times 2.$$

$$\textit{Moment of momentum on delivery} = \frac{2 \times 2240}{60 \times 32} \times 4 \times 3\tfrac{1}{2}.$$

$$\text{Sum} = Couple = 98.$$

$$HP = \frac{98 \times 2\pi \times 200}{60 \times 550} = 3.7.$$

TURBINES.

229. The general definition of a turbine is a water motor rotating about a vertical axis; but it is usually restricted to those horizontal wheels in which the water is conducted to the vanes by curved guide-plates. They may be divided into four classes:

1. *Outward flow turbines*, in which the water flows horizontally outward from the central shaft, and is discharged at the outer circumference of the wheel.

2. *Inward flow turbines*, in which the water is admitted at

the outer circumference and flows horizontally inward toward the central shaft.

3. *Parallel flow turbines*, in which the water flows downward (or upward) through the wheel.

4. *Mixed flow turbines*, in which the flow may be compounded of the 1st or 2d with the 3d above.

230. FOURNEYRON'S TURBINE, invented about the year 1827, is a good type of an outward flow turbine. The central shaft

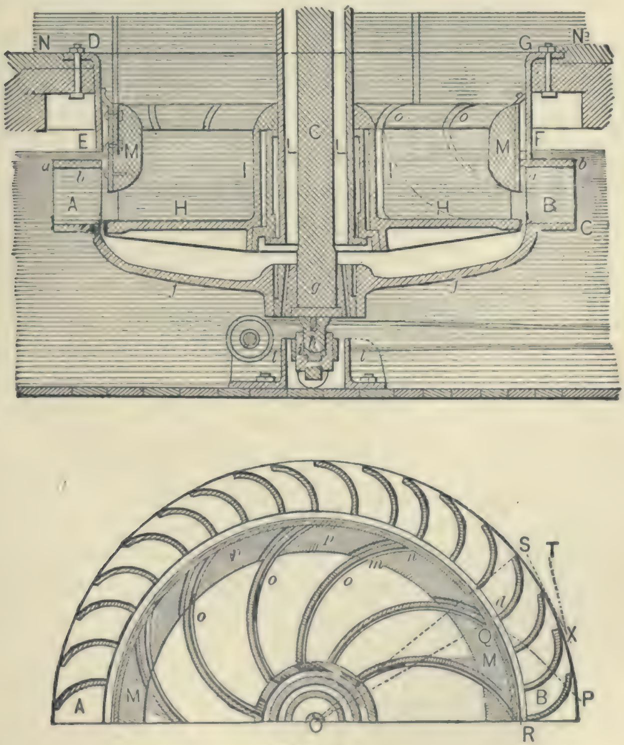

FIG. 177.

C of the wheel, to which the driving mechanism is attached, passes down through the supply chamber *GH*, from which it is separated by the tube *LL*.

The water passes downward between the guide plates *oo*, and outward between the same, through the gates *MM*, thence against and along the vanes *AB*, finally escaping at the outer circumference of the wheel. The vanes curve backward so as to cause the water to be discharged in a direction opposite to that of the wheel, relatively to the vanes. The vanes are attached to the shaft *C* by means of the rigid arms *ff*.

To find the mechanical power expended upon the wheel, let, in Fig. 177,

v = the velocity of the water as it enters the vanes,
$\gamma = PQR$ = the angle between the tangent PQ to the guide plate, and QR, the tangent to the inner rim of the wheel,
v_t = the tangential component of the velocity,
v_r = the radial component of the velocity,
$r_1 = OQ$ = the internal radius of the wheel,
α = angle between the direction of the water as it escapes from the wheel, and XS the tangent to the outer rim of the wheel,
v', v_t', v_r', r_2 = respectively, the velocity of discharge, its tangential and radial components, and the radius of the outer rim.

Then,

$$v_t = v \cos\gamma, \qquad v_r = v \sin\gamma; \tag{375}$$

$$v_t' = v' \cos\alpha, \qquad v_r' = v' \sin\alpha. \tag{376}$$

The moment of the momentum of the radial velocities will be zero; hence, the moment of the momentum of the water as it enters the wheel will be

$$Mr_1v_t,$$

and as it quits it, it will be

$$Mr_2v_t';$$

hence, Article 227, the work imparted to the wheel, neglecting friction, contractions, eddying, etc., will be

$$Pu = M(r_1v_t - r_2v_t')\varpi; \tag{377}$$

and if h be the head in the supply chamber above the point of discharge, the efficiency will be

$$e = \frac{(r_1 v_t - r_2 v_t')\,\omega}{gh}. \tag{378}$$

Equation (377) is a fundamental one in the theory of turbines. The velocities v and v' will generally be independent of each other, depending upon the form of the guide plates, the vanes, and the speed of the wheel.

For the highest efficiency, the water must quit the wheel with no velocity. There will, however, practically, always be a radial velocity, but when this is reduced to a minimum by the proper construction of the vanes, a *maximum* efficiency will be secured by running the wheel at such a speed as to make the velocity of discharge a minimum. Assuming it a minimum when the discharge is radial (approximately true), we have

$$v_t' = 0, \tag{379}$$

$$Pu = Mr_1 v_t \omega,$$

$$e = \frac{r_1 v_t \omega}{gh}. \tag{380}$$

The coefficients of M, equations (377) and (379), divided by g, are the *effective heads* for the respective cases. If to the effective head there be added the head due to the loss of velocity, there being no frictional resistances, the sum will be the head in the supply chamber. The only head lost in the case of maximum efficiency will be that due to the radial velocity of discharge. For this case, α, equation (376), will be $90°$, and $v_r' = v'$. But if w be the velocity of the water along and relatively to the vane, and w_t and w_r the relative tangential and radial velocities, respectively, $\beta =$ the angle SXT, Fig. 177, between the tangents respectively to the outer rim of the wheel and the vane; then

$$\left.\begin{aligned} w_t = w \cos\beta;\ w_r = w \sin\beta; \\ w_r = w_t \tan\beta; \end{aligned}\right\} \tag{381}$$

the head corresponding to which will be

$$w_i^2 \tan^2 \beta \div 2g;$$

hence, for the case of maximum efficiency,

$$h = (2r_1 v_i \omega + w_i^2 \tan^2 \beta) \div 2g; \tag{382}$$

from which we find that the velocity with which the water enters the wheel is

$$v = v_i \sec \gamma = \left(\frac{2gh - w_i^2 \tan^2 \beta}{2r_1 \omega}\right) \sec \gamma. \tag{383}$$

Hence the velocity of entering the wheel varies inversely as the velocity of the inner rim.

As limiting cases assume that $\gamma = 0$, and $\beta = 0$, and we have

$$v = \frac{gh}{r_1 \omega}. \tag{384}$$

If $r_1 \omega = v$, then

$$v^2 = gh, \tag{385}$$

or the velocity will be that due to one-half the entire head.

If $r_1 \omega = \frac{1}{2} v$, then

$$v^2 = 2gh, \tag{386}$$

or the velocity will be that due to the head.

If $r_1 \omega = \frac{1}{4} v$,

$$v^2 = 4gh,$$

or the velocity will be that due to twice the head, which is an hypothetical condition.

The above analysis is equally applicable to the other classes of turbines; but the construction must be different for each class in order to realize the conditions here imposed. Thus it will be found that β must be larger for a parallel-flow turbine than for an outward flow, and still larger for an inward flow, if the transverse sections of the passages through the wheel are uniform. Its value may also depend upon r_1 and r_2, as will be shown.

231. Special Cases.—**1.** In the Fourneyron turbine the initial elements of the guide plates are radial, and passing outward from the axis of the supply chamber, are gradually curved until they finally terminate as nearly tangent to the inner rim of the wheel as practicable, allowing sufficient space between the plates for the passage of the required quantity of water. The least value of γ,

$$\tan \gamma = \frac{v_r}{v_t}, \tag{387}$$

gives the greatest possible tangential velocity to the water when it enters the wheel.

The initial elements of the vanes are also radial, but in passing outward they are curved backward, and at their terminus make a small angle, β, with the outer rim. These conditions give

$$\tan \beta = \frac{w_r}{w_t}. \tag{388}$$

In order that the water shall enter without impulse, the velocity of the inner rim of the wheel must equal the tangential velocity of the water, or,

$$v_t = r_1\omega; \tag{389}$$

in which case the initial velocity of the water relatively to the wheel will be the radial velocity v_r. It will then pass along the curved vanes and be finally discharged backward relatively to the vane with a tangential velocity w_t, but the vane has a forward motion of $r_2\omega = nr_1\omega$; hence the actual tangential velocity of the water will be

$$v_t' = w_t - nr_1\omega, \tag{390}$$

which for the best result must be zero, for which condition we find, combined with equations (388), (389), and (387),

$$w_t = nr_1\omega = \frac{w_r}{\tan \beta} = nv_t = \frac{nv_r}{\tan \gamma}, \tag{391}$$

$$\therefore \tan \gamma = n\frac{v_r}{w_r} \tan \beta; \tag{392}$$

which establishes the relation between the outer angles of the blades and vanes relatively to the respective rims of the wheel. The radial velocity through the wheel may be made constant when the wheel passages are full, centrifugal force being neglected, by making the transverse sections of these passages uniform; and since their breadths vary as the radii r_1 and r_2, the depths must vary inversely as the same numbers, thus making the axial sections hyperbolas. This being done, $v_r = w_r$, and we have

$$\tan \gamma = n \tan \beta, \tag{393}$$

which is the condition commonly assumed. These conditions being realized, we have for the Fourneyron type—frictional losses being abstracted—equations (380), (382), (389), and (390),

$$e = \frac{2}{2 + n^2 \tan^2 \beta}, \tag{394}$$

$$v_i^2 = \frac{2\,gh}{2 + n^2 \tan^2 \beta}, \tag{395}$$

and from (379),

$$Pu = \frac{62\frac{1}{2}}{32\frac{1}{6}}\,Qv_i^2 = Wh\frac{1}{1 + \frac{1}{2}n^2 \tan^2 \beta}; \tag{396}$$

from which it appears that the efficiency is greatest when β is least.

If $\beta = 0$; then $e = 1$ and $v = \sqrt{gh}$; and the head due to the velocity with which the water enters the wheel is one-half that due to the head in the supply chamber. Since the water enters the wheel with a small velocity relatively to the vanes, $v_i \tan \gamma$, and is discharged with a greater relative velocity, $w = nr_1\omega \sec \beta$, it follows that from the point of entrance the velocity of the water *relatively to the vane* is accelerated to the point of discharge, while the *actual* velocity diminishes along the same path.

2. In the *cup vane turbine* the guide plates are essentially the same as the Fourneyron type; but the water issues from them

with a velocity due to the head in the supply chamber, and hence enters the wheel as a jet. The vanes are vertical, and their surfaces are as nearly semi-cylindrical as the circumstances of construction will permit. To avoid loss from shock, the initial elements of the vanes should be tangential to the direction of the water *relatively to the vanes* as it enters them.

Let γ' be the angle between the initial element of the vanes and the inner rim of the wheel, v' the velocity relative to the vane at entrance and which will be made uniform by making the normal sections of the stream uniform, and V the actual velocity of discharge, then

$$v \cos \gamma - v' \cos \gamma' = r_1 \omega,$$

$$v' \sin \gamma' = v \sin \gamma,$$

$$n r_1 \omega - v' \cos \beta = \text{tang. vel. at exit},$$

$$v' \sin \beta = \text{radial vel. at exit},$$

$$V = \left[(n r_1 \omega - v' \cos \beta)^2 + v'^2 \sin^2 \beta \right]^{\frac{1}{2}},$$

which should be a minimum for maximum efficiency. Assume either γ or γ' as known, v being known, and find V a function of ω, and the value of ω that will make V a maximum will make known γ or γ', and v'. Instead of reducing this case, we will take a special one, assuming that the velocity of the inner rim of the wheel is equal to one-half the tangential velocity of the water, and that the discharge from the wheel is radial; then

$$r_1 \omega = \tfrac{1}{2} v \cos \gamma, \tag{397}$$

$$v_r = v \sin \gamma, \tag{398}$$

$$\tan \gamma' = \frac{v \sin \gamma}{\frac{1}{2} v \cos \gamma} = 2 \tan \gamma, \tag{399}$$

$$v' = \tfrac{1}{2} v \sqrt{\cos^2 \gamma + 4 \sin^2 \gamma}, \tag{400}$$

$$n r_1 \omega = v' \cos \beta; \tag{401}$$

for which case β cannot be assumed, but may be found from the last equation.

The energy imparted to the wheel will be, equations (379), (397),

$$Pu = Mv_t \cdot r_1 \omega = Mv \cos \gamma \cdot \tfrac{1}{2}v \cos \gamma,$$

$$= \tfrac{1}{2}Mv^2 \cos^2 \gamma. \qquad (402)$$

The kinetic energy of the water as it leaves the gates will be, W being the weight of water flowing per second,

$$Wh = \tfrac{1}{2}Mv^2; \qquad (403)$$

hence the theoretical efficiency will be

$$e = \frac{\tfrac{1}{2}Mv^2 \cos^2 \gamma}{\tfrac{1}{2}Mv^2} = \cos^2 \gamma. \qquad (404)$$

3. In the *inward flow turbine* the initial elements of the guide curve are at the outer circumference of the supply chamber, and curve so as to form a small angle with the outer rim of the wheel. The outer elements of the vanes are radial, and as they approach the inner rim of the wheel, they curve in a direction opposite to that of the guide plates. Since n, in equation (393), will be fractional, the angle β will exceed γ. The water should enter the wheel with a tangential velocity equal to the velocity of the outer rim. The remainder of the analysis is the same as for the Fourneyron turbine. If the vanes are cup-shaped, the analysis will be the same as the second case above.

4. In the *parallel flow turbine*, the initial elements of the guide plates are vertical, and are curved as they pass downward so as to terminate as nearly horizontal as possible. The initial elements of the vanes are usually vertical, and the vanes curve in an opposite direction from that of the guide plates, terminating at the same angle with the horizontal. This last is a result of making $n = 1$ in equation (393), giving

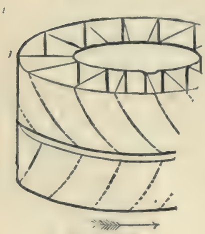

FIG. 178.

$$\gamma = \beta.$$

5. If, in an *outward flow turbine*, the vanes make an angle with the inner rim, exceeding 90° and less than 180°, which will be an angle greater than for the Fourneyron type, and less than for the cup vane, we shall find

$$r\omega > \sqrt{gh} \text{ and } < \sqrt{2gh},$$

for a speed of maximum efficiency.

6. An unlimited number of turbines having different forms of vanes may be devised which will give, theoretically, high efficiencies; but if friction, whirls, contractions, etc., be involved, those forms will be, practically, most efficient, which for the same theoretical efficiency, have the least of the above named resistances.

232. GENERAL CASE.—In any hydraulic motor, let H be the total available head, h the loss of head due to location and form of wheel, or that portion of the head below the wheel (and possibly above) not utilized, h' the loss by shock, h'' the head lost on account of the velocity of the water at quitting the wheel, h''' the head lost from contractions, whirls, friction, etc., then will the mechanical power be

$$62\tfrac{1}{2}Q\,[H - h - h' - h'' - h'''] ; \tag{405}$$

and the gross efficiency of the motor will be

$$e = \frac{H - h - h' - h'' - h'''}{H}. \tag{406}$$

The efficiency of the motor as a machine may exceed this, for it may be so constructed and placed that its entire efficiency will be developed with a less head than H; as, for instance, in the case of an overshot wheel, if there were 4 or 5 feet fall below the lowest point of the wheel, it would in no way affect the efficiency of the wheel, but the wheel would fail to utilize the power of the water. In determining the quality of the wheel as a motor, only so much of the head should be

considered as is actually necessary for operating the wheel in the place where it is established.

EXAMPLES.

1. A stream delivers 10,000 gallons of water per minute to a Fourneyron turbine. If the fall be 100 ft., $\tan\beta = \frac{1}{3}$, $n = 1.4$, and revolutions per minute $= 240$; find the internal and external radii for the most efficient speed, the HP, and the depth of the casing at the outer periphery.

From equation (395) we find

$$v_t = 8\pi r_1 = \sqrt{\frac{2 \times 32\frac{1}{6} \times 100}{2 + (1.4)^2 \times \frac{1}{3}}},$$

or

$$r_1 = 2.14 \text{ feet.}$$

$$r_2 = nr_1 = 2.996 \text{ feet.}$$

From (394),

$$efficiency = \frac{2}{2 + (1.4)^2 \times \frac{1}{3}},$$

$$= .90.$$

$$HP = \frac{10,000 \times 231 \times 62\frac{1}{2} \times 100 \times .9}{1,728 \times 33,000},$$

$$= 227.8.$$

From (391),

$$w_r = v_r = nr_1\omega \tan\beta,$$

or

$$v_r = 25.1328 \text{ feet per second,}$$

and

$$d = \frac{10,000 \times 231 \times 12}{1,728 \times 25.133 \times 6\pi \times 60}$$

$$= .56 \text{ inches.}$$

2. An outward flow turbine is supplied with water having a head of 350 feet. The internal diameter is 15 inches, and the external diameter is 21 inches. Angle of guiding plates,

$\gamma = 16°$, and the vanes of the wheel are placed at the corresponding angle. Depth of wheel at the inner periphery, 2 inches. Find the number of revolutions per minute required for the best efficiency, neglecting frictional resistances.

Taking account of frictional resistances, and supposing the true efficiency to be $\frac{3}{4}$ths, find the best number of revolutions and HP developed.

From equation (395),

$$v_t = \sqrt{\frac{2 \times 32\frac{1}{6} \times 350}{2.0822}} = 104 \text{ feet per second.}$$

$$\frac{104 \times 60}{2\pi \times \frac{5}{8}} = 1{,}589 \text{ revolutions per minute.}$$

If we take frictional resistances into account, and suppose the wheel to be running at its greatest efficiency, we will have for the total energy,

$$mgQh = mQv_t^2 + \frac{mQ}{2} v_t^2 n^2 \tan^2 \beta + \frac{mQ}{2} \Sigma F v_t^2 (1 + n^2 \tan^2 \beta).$$

Therefore the efficiency is

$$e = \frac{2}{2 + n^2 \tan^2 \beta + \Sigma F (1 + n^2 \tan^2 \beta)},$$

$$v_t^2 = \tfrac{3}{4} gh, \quad \text{or} \quad v_t = 91.9 \text{ feet per second,}$$

$$n_1 = \frac{91.9 \times 60}{2\pi \times \frac{5}{8}} = 1{,}440 \text{ revolutions per minute,}$$

$$w_r = v_r = v_t \tan \gamma,$$

or

$$w_r = 26.35 \text{ feet per second.}$$

Area of flow at inner periphery = .6545 square feet:

$$HP = \frac{.6545 \times 26.35 \times 350 \times 62.5 \times 60 \times .75}{33{,}000}$$

$$= 514.45.$$

3. Which requires the larger gate opening for the same work, the Fourneyron turbine of Case 1, or the cup vane turbine of Case 2?

4. If the head in the supply chamber be 15 feet, find the velocity of the inner rim for best efficiency in Cases 1 and 2, when $\beta = 0$, and when $\beta = 20°$.

5. Explain why the same quantity of water flowing through the cup vane as through the Fourneyron turbine produces the same work, when the velocities of the water on entering the wheel and also the velocities of the wheel are different.

RESISTANCES.

233. Sudden enlargements.—If a stream flows along a pipe having a sudden enlargement, it suffers a loss of velocity.

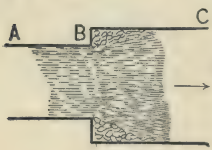

Fig. 179.

If v be the velocity in the small pipe, and v_2 that in the enlarged part, it might seem that the head lost would be $(v^2 - v_2^2) \div 2g$; but this is true only when the internal forces are functions of the distance between the centres of force, Article (151), a condition not realized in this case. The particles act upon each other according to a law not definitely known. It may be partly of the nature of friction, but chiefly it results in the production of *whirls* at the angles of the tube. It is also known that the reduction of the kinetic energy develops heat, and this, being retained in the tube, partially maintains the head, so that the loss is not as great as it otherwise would be. The action is not exactly of the nature of the impact of finite bodies, but resembles it in regard to the non-elasticity of the water and the comparatively sudden reduction of the velocity; and for these reasons we analyze the case as for inelastic impact. In all cases, however, when the law of action is unknown, the result should be checked by direct experiment. We may assume that the mass m passing out of the small tube in a unit of time impinges upon the mass M in the enlarged

part of the tube; then, according to equation (42), the loss of energy will be

$$\tfrac{1}{2}m\,(v - v_2)^2, \qquad (407)$$

and it is found that this expression represents with sufficient accuracy the results of experiment.

This amount of energy transformed into heat would raise a pound of water a number of degrees in temperature given by the expression

$$t^\circ = \frac{m\,(v - v_2)^2}{2 \times 772}, \qquad (408)$$

where 772 is Joule's mechanical equivalent of heat.

The head lost will be

$$\frac{(v - v_2)^2}{2g}. \qquad (409)$$

234. RESISTANCES IN LONG PIPES.—The principles of work and energy on which Bernoulli's theorem is founded may be extended to the case in which there is resistance along the stream between A and B, as is the case in actual tubes. The *law* of the resistance can be determined only by experiment. It has been found that, with sufficient accuracy, the resistance varies as the perimeter of contact between the liquid and pipe in a transverse section, called the *wetted perimeter*, also as the square of the velocity, and a factor f dependent upon the condition of the pipe. Hence, if R be the resistance at any section, s, w the weight of a unit of volume of the fluid, and c the wetted perimeter, we have

$$R = fwcr\frac{v^2}{2g}.$$

If the velocity and sections be uniform, then will the loss of head for a length l be

$$fl\frac{c}{s}\frac{v^2}{2g}. \qquad (410)$$

The value of f for rivers, as given by Eytelwein, is

$$f = a + \frac{b}{v}, \tag{411}$$

where

$$a = 0.007164, \quad b = 0.007409.$$

235. General case.—Let AB be a stream flowing from a reservoir whose upper surface is at C, and whose exit is at B. Or, to generalize it, let B be any point in the pipe. Through B draw a horizontal GB, then will $CG = BD$ be the total

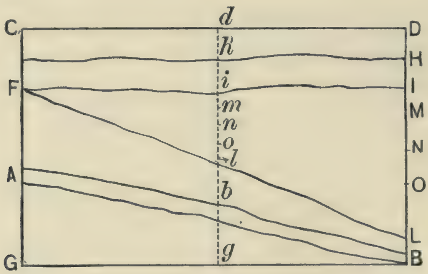

Fig. 180.

head on B, neglecting the pressure of the air as we may do in most practical cases. If B be the exit, let BL be the head necessary to overcome the pressure at the orifice. If the opening be the full size of the pipe, BL will be zero; but if it be contracted, it will be of finite value. Lay off on BD, the following heads:

$DH = h_1 = \dfrac{v^2}{2g} =$ the head due to the velocity of exit,

$HI = h_2 =$ the head lost due to the resistance of entering the pipe at A,

$IM = h_3 =$ the head necessary to overcome the resistances along the straight portions of the pipe,

$MN = h_4 =$ head lost by bends in the pipe,

$NO = h_5 =$ head lost by sharp angles,

$OL = h_6 =$ head lost by enlargements and contractions,
$LB = h_7 =$ head due to pressure at B,
$DB = H =$ total head at B.

Then

$$H = h_1 + h_2 + h_3 + h_4 + h_5 + h_6 + h_7. \quad (412)$$

The values of these several heads, with the exception of the first, are determined by experiment. They are usually determined as multiples of $h_1 = \frac{v^2}{2g}$, and hence may be written with the use of corresponding subscripts, thus

$$H = (1 + f_2 + f_3 + f_4 + f_5 + f_6 + f_7)\frac{v^2}{2g}. \quad (413)$$

The following are some of the principal values of the coefficients as determined by experiment:

Orifice in a thin plate, which would be applicable when the length of the pipe is zero,

$$f_2 = 0.054.$$

Straight, short cylindrical mouth-piece perpendicular to the side of the vessel,

$$f_2 = 0.505.$$

For the resistance along the pipe whose length is l, section s, wetted perimeter c,

$$f_3 = f\,l\frac{c}{s}.$$

Where f is dependent upon the dimensions of the pipe, and according to Darcy, its value is

$$f = k\left(1 + \frac{1}{d}\right).$$

Where d is the internal diameter of the pipe in inches, and k for clean cast-iron pipe, is

$$k = 0.005;$$

and for pipe very rough from sediment,

$$k = 0.01.$$

According to Weisbach, for clean cast-iron pipe,

$$f = 0.0036 + \frac{0.0043}{\sqrt{v}},$$

where v is the mean velocity. The larger computed value of f would ordinarily be used.


FIG. 181.

For a bend through an angle i, the radius of the curve being r, diameter of the pipe d,

$$f_4 = \frac{i}{\pi}\left\{0.131 + 1.847\left(\frac{d}{2r}\right)^{\frac{7}{2}}\right\}.$$

For a knee, or sharp angle, i,

$$f_5 = 0.946 \sin^2 \frac{i}{2} + 2.05 \sin^4 \frac{i}{2},$$

and if the knee is 90°, this becomes

$$f_5 = 1 \text{ nearly.}$$

The value of f_6 may be computed approximately, and f_7 is dependent upon the condition of the opening at exit.

These values in (413) enable one to determine v; or v and the other quantities being known, H, the necessary head, may be determined. Indeed any one of the many quantities may be found in terms of the others.

To find the pressure on the pipe at any point, as b, lay down from CD the head dh due to the velocity at b, hi the head lost at entrance A, im the head lost by the resistance of the straight part of the pipe from A to b, mn the head lost by

bending between A and b due to bends, no that lost by angles, ol all other heads lost; then will

$$lb = \textit{pressure on pipe at } b.$$

The head bg is potential in reference to BG. The line FL limits the upper ends of the heads due to pressure on the pipe; and if the pipe lay along this line there would be no pressure except on the under side due to the weight of the fluid. If an orifice be made at b, the liquid would flow out in a stream rising to the height bl, less the resistance at exit from said orifice and the resistance of the air. But if the pipe lay along FL, the liquid would not flow out of an orifice at l.

Examples.

1. Let it be required to deliver water with a given hydraulic head h_0, such, for instance, as is necessary to drive an engine. Then

$$H - h_0 = \frac{v^2}{2g} + \Sigma F \frac{v^2}{2g} + 4f \frac{l}{d} \frac{v^2}{2g},$$

where ΣF is the sum of the coefficients due to valves, bends, knees, sharp-edged entrance, etc., and f that for surface friction. Therefore we have

$$\frac{v^2}{2g} = \frac{H - h_0}{1 + \Sigma F + 4f\frac{l}{d}},$$

from which the velocity is obtained.

If Q be the required quantity, and S the section, then

$$S = Q \div v.$$

2. Water flows from a tank through a 1 in. vertical pipe. Find the head in the tank so that the velocity of discharge may be the same for every length, taking into account the resistance of sharp-edged entrance as well as of surface friction.

Let l be the length of the pipe, and H the head above the orifice; then

$$H + l = \left(1 + f_2 + \frac{flc}{S}\right)\frac{v^2}{2g},$$

or

$$H = (1 + f_2)\frac{v^2}{2g} + \left(\frac{v^2}{2g}\frac{fc}{S} - 1\right)l.$$

But the conditions require that the term containing l shall disappear; hence

$$\frac{v^2}{2g} \cdot \frac{fc}{S} = 1,$$

and

$$H = (1 + f_2)\frac{v^2}{2g},$$

where v is independent of l as required.

Then

$$\frac{v^2}{2g} = \frac{S}{fc},$$

and

$$H = (1 + f_2)\,\frac{S}{fc}$$

$$= \frac{d}{4 \times .005\left(1 + \frac{1}{d}\right)}(1 + \tfrac{1}{2})$$

$$= 3.7 \text{ feet}.$$

3. Water flows from a tank through a uniform sloping pipe of diameter d. Taking into account the resistance at entrance, show that the water will flow with the same velocity whatever be the length of pipe, if the pipe slopes at an angle, $\sin\theta = \frac{.04}{3}\left(\frac{d+1}{d^2}\right) H_0$, H_0 being the head in the tank above the entrance.

We have, as before,

$$H_0 + l \sin \theta = \frac{v^2}{2g}\left(1 + F + f\frac{l}{m}\right).$$

We will suppose θ so small that the resistance to sharp-edged entrance may be taken as $F = \frac{1}{2}$. We have $f = .005\frac{d+1}{d}$, and $m = \frac{d}{4}$. Hence we have

$$H_0 + l \sin \theta = \frac{v^2}{2g}(1 + F) + \frac{v^2}{2g}4f\frac{l}{d},$$

which may be regarded as an equation of identity in l. Therefore equating like terms,

$$H_0 = \frac{v^2}{2g}(1 + F),$$

or

$$\frac{v^2}{2g} = \frac{H_0}{1.5}.$$

Also

$$\sin \theta = \frac{v^2}{2g}\frac{4f}{d}$$

$$= \frac{.04}{3}\frac{(d+1)}{d^2}H_0.$$

GASES.

236. The density of an ideal, incompressible fluid is independent of the pressure to which it is subjected, and dependent only upon its constitution; but the density of a compressible fluid is dependent upon both its constitution and external pressure. Thus, if water were strictly non-compressible, its density at all depths in the ocean would be the same as at the surface; but the atmosphere, being a gas, diminishes in density as we ascend, since the weight of the atmosphere above

any point is less the higher the point. The laws of the pressure of fluids given in Articles 174 to 178 inclusive are applicable to gases. But if the pressure varies, the density and temperature both vary according to laws which are determined by experiment.

237. Boyle's (or Mariotte's) Law.—According to the experiments of Boyle and Mariotte, the volume of a gas *at uniform temperature* varies inversely as the pressure to which it is subjected. Hence, if v_0 be a known initial volume of gas, and p_0 the initial pressure per unit of area to which the gas is subjected—being the pressure upon its bounding surface, or at any point within it, for which latter reason it is also called *the tension* of the gas—and v and p any other contemporaneous volume and pressure; then we have

$$pv = p_0v_0 = constant = m, \tag{414}$$

which, if p and v vary simultaneously, is the equation of an equilateral hyperbola referred to its asymptotes. In testing this law experimentally, it is necessary, after compressing or dilating the gas, to make the temperature of the gas the same as the initial temperature; in the former case cooling, and in the latter heating it. The law is found to be *very nearly*, but not exactly, correct for air and other known gases, the agreement being nearer the more perfect the gas, from which it is inferred that it correctly represents the law for the ideally perfect gas.

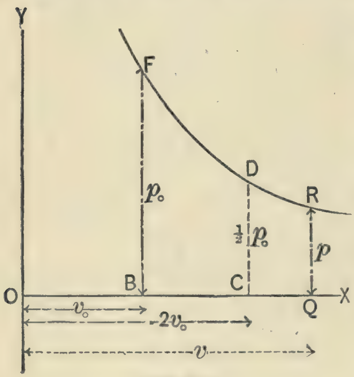

Fig. 182.

Since the density, δ, varies inversely as the volume, we have, *for uniform temperature*,

$$\delta v = \delta_0 v_0 = constant, \tag{415}$$

and

$$\delta p_0 = \delta_0 p. \tag{416}$$

238. To find the tension at any point of a column of gas of uniform temperature.—On account of the weight of the gas and its compressibility, its density will vary as some function of the height. Conceive a prismatic column of gas whose base is unity, and at the lower base p_0 the tension, δ_0 the density, w_0 the weight of a unit of volume, and at height z let the corresponding quantities be p, δ, w. Then will the pressure on the lower base equal the pressure at the height z added to the weight of the prism of gas of that height; or,

$$p_0 = p + \int_0^z w dz,$$

and differentiating,

$$dp = -wdz, \tag{417}$$

which is independent of the *law of pressure* and of the total head.

According to Mariotte's law,

$$\delta : \delta_0 :: \frac{w}{g} : \frac{w_0}{g_0} :: p : p_0, \tag{418}$$

and considering gravity as uniform,

$$w = \frac{w_0}{p_0} p. \tag{419}$$

Substituting and reducing,

$$\frac{dp}{p} = -\frac{w_0}{p_0} dz;$$

integrating between the limits p and p_0, z and 0, gives

$$\log \frac{p}{p_0} = -\frac{w_0}{p_0} z;$$

$$\therefore p = p_0 e^{-\frac{w_0}{p_0} z}. \tag{420}$$

Similarly,

$$w = w_0 e^{-\frac{w_0}{p_0}z}; \quad \delta = \delta_0 e^{-\frac{g_0 \delta_0}{p_0}z}. \qquad (421)$$

The weight of a prism of gas of height h, will be

$$W = \int w dz = w_0 \int_0^h e^{-\frac{w_0}{p_0}z} dz = p_0 \left[1 - e^{-\frac{w_0}{p_0}h} \right]. \qquad (422)$$

If $h = \infty$, $W = p_0$, as it should.

239. Numerical values. — The mean pressure of the atmosphere at the level of the sea is $p_0 = 14.7$ pounds per square inch, or 2116.8 pounds per square foot.

The weight of a cubic foot of pure dry air under the pressure of 14.7 pounds per square inch, and at the temperature of melting ice (32° F.),

$$w_0 = 0.080728 \textit{ pound avoirdupois.} \qquad (423)$$

If the atmosphere were pure, dry, and uniform, and of the density as at present at the level of the sea where the temperature is 32° F., the height would be

$$H = \frac{2116.8}{0.080728} = 26{,}221 \text{ feet}; \qquad (424)$$

or the height of one atmosphere of uniform density is nearly 5 miles; but as the atmosphere does not fulfil these conditions, and being a little heavier, the height is a little less than this value.

At the level of the sea the barometer stands at 29.92 inches nearly.					
At 5,000 feet above "	"	"	24.7	"	"
At 10,000 feet (Mt. Ætna)	"	"	20.5	"	"
At 15,000 feet (Mt. Blanc)	"	"	16.9	"	"
At 3 miles	"	"	16.4	"	"
At 5 miles	"	"	8.9	"	"

240. To find the tension at extreme heights when the variation of gravity is considered.

From equation (418) we have

$$w = \frac{w_0}{p_0} \cdot \frac{g}{g_0} p. \tag{425}$$

Let R be the radius of the earth, z any distance above the earth, then

$$g = g_0 \frac{R^2}{(R+z)^2}; \tag{425a}$$

and (417), (425), (425a), give

$$\frac{dp}{p} = -\frac{w_0}{p_0} \frac{R^2}{(R+z)^2} dz;$$

integrating between the limits of p and p_0, z and **0**,

$$\log \frac{p}{p_0} = -\frac{w_0}{p_0} \frac{Rz}{R+z};$$

$$\therefore p = p_0 e^{-\frac{w_0}{p_0} \frac{Rz}{R+z}};$$

and similarly,

$$\delta = \delta_0 e^{-\frac{w_0}{p_0} \frac{Rz}{R+z}}.* \tag{425b}$$

Assuming the density of air at the earth $\delta_0 = \frac{1}{400}$ of a pound per cubic foot, as it is very nearly, and $R = 20{,}860{,}000$ feet, and equation (425b) reduces to

$$\delta = \tfrac{1}{400}(10)^{-345 \frac{z}{R+z}}. \tag{425c}$$

If $z = \infty$, we have

$$\delta = \frac{1}{400 \times (10)^{345}}, \tag{425d}$$

which is the limit of the density. If gravity be considered uniform, we would have for the height equal to the radius of

* A more cumbersome demonstration of this formula may be made by means of Prop. xxii., B. II., of Newton's *Principia*.

the earth, $z = 20{,}860{,}000$, and (421) and (418) reduce to the same as (425d).

241. HEIGHTS may be determined approximately by the pressure of the atmosphere. If its temperature were uniform and free from currents, we would have from equation (420),

$$z = \frac{p_0}{w_0} \log \frac{p_0}{p} = 26{,}221 \log_\epsilon \frac{b_0}{b},$$

where b_0 and b are the readings of a barometer at the lower and higher stations respectively. Adapting this to the use of common logarithms, we have

$$z = 26{,}221 \times 2.30258 \log \frac{29.92}{b}$$

$$= 60{,}735 \log \frac{29.92}{b}.$$

But this coefficient is known to be too large, and practice shows that

$$z = 60{,}345 \log \frac{29.92}{b} \tag{426}$$

gives better results.

If z_1 be the height of another station, then

$$z_1 = 60{,}345 \log \frac{29.92}{b_1};$$

from which subtracting the former, we have

$$z_1 - z = h = 60{,}345 \log \frac{b}{b_1}. \tag{427}$$

This result, however, requires to be corrected for the effect of the temperature on the mercurial column, the effect of the variation of gravity due to great heights, change of gravity due to latitude, the reduction of observations when not simultaneous, the effect upon the mercurial column due to the

attraction of the mountain on which observations are made, all of which are well discussed by Poisson in his *Traité de Mécanique.*

If T = the temp'ture of attached thermometer at lower station,
T' = " " " upper "
t = " detached " lower "
t' = " " " upper "
φ = the latitude of the place;

then,

$$h = 60{,}345 \frac{1 + 0.00102\,(t + t' - 64)}{1 - 0.002551 \cos 2\varphi}$$

$$\times \log_{10} \left[\frac{b}{b_1} \cdot \frac{1}{1 + 0.0001001\,(T - T')}\right] \text{ feet.} \qquad (427a)$$

Observations should be taken at both stations simultaneously, in calm weather; but when this cannot be done, two observations may be made at one station at different hours, and one at the other at a time midway between, and the mean of the former be used as one observation.

242. Gay Lussac's (or Charles') Law.—It was found by these men by direct experiment, that the increase of tension in a fixed volume of gas is directly proportional to the increase of temperature.

If p_0, t_0, v_0, be respectively the initial pressure, temperature, and volume of a gas, and p, t, v, any other corresponding contemporaneous values, and β a coefficient of expansion, then we have, the pressure being constant,

$$v = v_0\,(1 + \beta\,(t - t_0)), \qquad (428)$$

the volume being constant,

$$p = p_0(1 + \beta(t - t_0)). \qquad (429)$$

From equation (428),

$$\beta = \frac{v - v_0}{(t - t_0)v_0}. \qquad (430)$$

In practice, the initial temperature is taken as that of melting ice, 32° F., or 0° C., and the initial pressure that of one atmosphere, or 14.7 lbs. per square inch, or 29.92 inches of mercury (760 millimeters). The initial volume may be of any convenient value. For these values it has been found that

for carbonic acid,	$\beta = 0.0037099$	for 1° C,
for air,	$\beta = 0.0036706$	"
for hydrogen,	$\beta = 0.0036613$	"

From these values it appears that the more perfect the gas, the less will be the coefficient of expansion; and Rankine concluded, in accordance with the theory of molecular vortices, that the limiting value is

$$\beta = .000365 = \tfrac{1}{274} \text{ for } 1^\circ \text{ C.}$$

$$= 0.0020275 = \frac{1}{493.2} \text{ for } 1^\circ \text{ F.,} \tag{431}$$

which values are now used for the *ideally perfect gas*, and are sufficiently accurate for air and several other of the more perfect gases.

243. Absolute zero. — Equation (429) becomes, for $t_0 = 0^\circ$ C.,

$$p = p_0\left(1 + \frac{t}{274}\right), \tag{432}$$

in which, if $t = -274^\circ$, p becomes zero, and a perfect gas would be destitute of tension. Similarly, according to (428), the volume would vanish. In Fahrenheit's scale t_0 is 32°, and equation (429) becomes

$$p = p_0\left(1 + \frac{t - 32}{493.2}\right), \tag{433}$$

in which, if $t = -461.2$, p becomes zero as before. According to the modern theory of heat, at this temperature the molecules of the gas would be at perfect rest. It is the point of absolute deprivation of heat, and is called *the zero of abso-*

lute temperature. No such temperature can even be approximated by any known process, the lowest recorded temperature obtained by artificial means being $-140°$ C. ($-284°$ F.); but an extension of the law of uniform expansion leads to this result. Temperatures reckoned from the absolute zero are called *absolute temperatures*, and are especially useful in simplifying many formulas in regard to heat. If t be the temperature on Fahrenheit's scale, and τ the same temperature on the absolute scale, $\tau_0 = 493.2$, the temperature of melting ice, then

$$\tau = 461°.2 + t = \tau_0 - 32° + t,$$

and equation (433) becomes

$$p = p_0 \frac{\tau}{\tau_0}. \tag{434}$$

If w be the weight of a cubic foot of dry air at pressure p pounds per square inch, and at absolute temperature τ, then

$$w = w_0 \frac{p}{p_0} \frac{\tau_0}{\tau}; \tag{435}$$

and if w' be the weight at temperature τ' and tension p', then

$$w' = w \frac{p'}{p} \frac{\tau}{\tau'}; \quad \delta = \delta_0 \frac{p}{p_0} \cdot \frac{\tau_0}{\tau}; \tag{436}$$

and since for gravity uniform, the volume varies inversely as the weight,

$$v = v_0 \frac{p_0}{p} \cdot \frac{\tau}{\tau_0}. \tag{437}$$

Since the mechanical properties of gases, whether at rest or in motion, involve the property of heat, we now consider some of the abstract properties of the latter.

244. Heat *is a form of energy.* It is not force, since force is only one of the elements producing energy. See Articles 25, 26, 151. It is believed to be a certain manifesta-

tion of the motion of the particles of a body.* The following principles, the result of observation and experience, have become established.

1. Heat may be transformed into external work; and conversely, work, under certain conditions, may be transformed into heat.

2. Heat cannot be transferred from a body of a lower to one of a higher temperature except by the aid of a machine and the expenditure of mechanical energy.

245. THE THERMAL UNIT is the amount of heat energy necessary to raise a unit of weight of ice-cold water one degree on the thermometric scale. The *English thermal unit* is the amount of heat energy necessary to raise one pound of water from the temperature of °32 F. to 33° F.; the value of which, as found by Joule, is the heat produced by friction in bringing a body weighing one pound to rest after falling 772 feet in a vacuum.† Or, more briefly, the English thermal unit equals 772 foot-pounds of work. This is called Joule's equivalent, and is usually represented by J.

The equivalent in French units is the heat energy necessary to raise one kilogramme of water from 0° C. to 1° C., and equals 424 kilogramme-metres.

The amount of heat energy necessary to raise the temperature 1° is not the same from all temperatures, although for ordinary mechanical purposes it is considered constant for the same substance.

246. SPECIFIC HEAT is the heat necessary to raise a unit of weight of any substance one degree on the thermometric scale, the thermal unit being unity.

Strictly speaking, *specific heat* is a ratio, being the quotient obtained by dividing the quantity of heat required to raise the temperature of one pound of the substance one degree by the quantity required to raise the temperature of one pound of water the same amount.

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* *Elementary Mechanics*, p. 69. Stewart on *Heat*.

† *Elementary Mechanics*, pp. 71, 72.

The specific heat under *constant pressure* of gases, is the specific heat determined when the gas is permitted to expand under the constant pressure. We denote it by c_p.

The specific heat under *constant volume* is the specific heat determined when the volume of the gas remains fixed. We denote it by c_v.

The former always exceeds the latter, for in addition to increasing the molecular motion of the gas, external work is done against the pressure of the air, expanding it about 0.0020275 of its volume for each degree, as shown in equation (431).

If the same quantity of heat will raise one pound of water m degrees, and one pound of any other substance n degrees, both under constant pressure, then

$$c_p = \frac{m}{n}.$$

The following are the values of c_p, c_v, and $c_p \div c_v$ for a few gases:

GAS.	c_p	c_v	$c_p \div c_v$
Air............	0.238	0.169	1.408
Oxygen........	0.218	0.156	1.400
Hydrogen......	3.405	2.410	1.412
Steam	0.480	0.370	1.297

247. THE DYNAMICAL SPECIFIC HEAT is the specific heat expressed in foot-pounds of work. If K_v be the dynamical specific heat under constant volume, and K_p that under constant pressure, then

$$K_v = Jc_v, \quad K_p = Jc_p; \tag{438}$$

where $J = 772$ foot-pounds, or 424 kilogramme-metres.

248. ADIABATIC CURVE.—Let q be the heat energy necessary to change the temperature of a unit of weight of a gas t degrees, the pressure being p, and density δ. Then

$$q = f(p \cdot \delta \cdot \tau). \tag{439}$$

Differentiating, p being constant, and reducing by the aid of (436), gives

$$c_p = \left(\frac{dq}{d\tau}\right)_p = \frac{d\delta}{d\tau} \cdot \frac{dq}{d\delta} = -\frac{\delta}{\tau} \cdot \frac{dq}{d\delta}; \tag{440}$$

and differentiating, considering δ as constant,

$$c_v = \left(\frac{dq}{d\tau}\right)_\delta = \frac{dp}{d\tau} \cdot \frac{dq}{dp} = \frac{p}{\tau} \cdot \frac{dq}{dp}. \tag{441}$$

Letting

$$\gamma = \frac{c_p}{c_v}, \tag{442}$$

γ being considered constant, (440) and (441) give

$$\delta \frac{dq}{d\delta} + \gamma p \frac{dq}{dp} = 0, \tag{443}$$

in which τ is involved only implicitly. Eliminating τ from (439) by means of (434), and

$$q = f(p \cdot \delta), \tag{444}$$

the total differential of which is

$$dq = \frac{dq}{dp} dp + \frac{dq}{d\delta} d\delta,$$

in which substitute $\frac{dq}{d\delta}$ from (443), and we find

$$dq = \frac{\gamma\delta}{p^{\frac{1-\gamma}{\gamma}}} \cdot \frac{dq}{dp} \cdot \frac{\frac{\delta p^{\frac{1-\gamma}{\gamma}}}{\gamma} dp - p^{\frac{1}{\gamma}} d\delta}{\delta^2}$$

$$= f\left(\frac{p^{\frac{1}{\gamma}}}{\delta}\right) d\left(\frac{p^{\frac{1}{\gamma}}}{\delta}\right);$$

$$\therefore q = F\left(\frac{p^{\frac{1}{\gamma}}}{\delta}\right);$$

$$\therefore p = \delta^\gamma \varphi(q); \tag{445}$$

where f is an arbitrary function, that part of which depending upon $\frac{dq}{dp}$ is implicitly a function of p and δ, as shown by

(444), and the other part, $\delta \div p^{\frac{1-\gamma}{\gamma}}$, is explicitly a function of the same quantities. The product of the two parts is some function of the same quantities expressed in terms of the ratio $p \div \delta$; the entire transformation being made so that an integral may be obtained, though it be in a functional form. F represents some other arbitrary function, and φ the inverse of it.

In the particular case where *q remains constant*, as it would when a gas is compressed or dilated in a close vessel impervious to the transmission of heat through its walls, although the indicated temperature would change, we would have for another pressure p_0, and density δ_0,

$$p_0 = \delta_0^\gamma \varphi(q);$$

and eliminating $\varphi(q)$ between these equations, we have

$$p\delta_0^\gamma = p_0\delta^\gamma; \tag{446}$$

and since the volumes will be inversely as the densities, we also have

$$pv^\gamma = p_0 v_0^\gamma = constant, \tag{447}$$

and from (436), (446), (447), we have

$$\frac{\tau}{\tau_0} = \left(\frac{\delta}{\delta_0}\right)^{\gamma-1} = \left(\frac{p}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{v_0}{v}\right)^{\gamma-1} = \left(\frac{w}{w_0}\right)^{\gamma-1}. \tag{448}$$

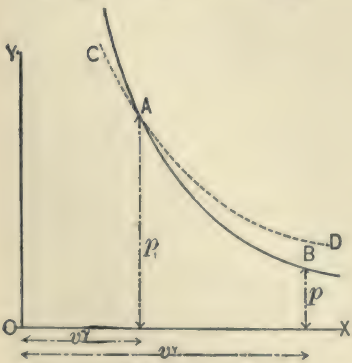

FIG. 183.

Equation (447) is the simplest form of the equation of the *Adiabatic curve*, or "curve of no transmission" (from the Greek α, not, and $\delta\iota\alpha\beta\alpha\acute{\iota}\nu\epsilon\iota\nu$, to pass through), and is represented by the curve AB. If an isothermal line pass through the point A, where the volume of the gas is 1 and pressure p_1, it

will pass above the adiabatic for values of $v > 1$, and under it for values of $v < 1$.

Examples.

1. If a given volume of air has a temperature of 85° F., what will be its temperature when dilated to double the volume, performing work, but receiving no heat?

We have, from equation (448),

$$\left(\frac{2v}{v}\right)^{0.408} = \frac{461.2 + 85}{461.2 + t};$$

$$\therefore\ t = -50^\circ \text{ F.}$$

2. If a prism of air having a tension of $1\frac{1}{2}$ atmospheres at a temperature of 88° F. expands adiabatically to a tension of one atmosphere, required the final temperature.

We have

$$\left(\frac{1}{1\frac{1}{2}}\right)^{0.2907} = \frac{461.2 + t}{549.2};$$

$$\therefore\ t = 24^\circ.35 \text{ F.}$$

3. Required the ratio of the volumes before and after expansion in the preceding example. *Ans.* 1.335.

4. If air be compressed adiabatically from a tension of 15 pounds at 50° F. to that of 90 pounds, required the final temperature.

5. If air at 60° F. and six atmospheres expands adiabatically to one atmosphere, required the final temperature.

249. Velocity of a wave in an elastic medium.* Assume

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* The general problem of wave propagation has received the attention of several of the most eminent mathematicians since the days of Newton, and many problems have been solved in a satisfactory manner. The simple method of Newton, *Principia*, Prob's XLIII.–L, B. II., has not been excelled, and the definite theoretical result obtained is quoted to the present day, although the effect of heat upon the velocity of sound was not then known. La Place, in the *Mécanique Céleste*, tomes II. and V., has treated of the oscillations of the sea and atmosphere; Lagrange, in the *Mécanique Analytique*,

that the medium is confined in a prismatic tube of section unity, E the coefficient of elasticity for compression, p a force which will produce a compression dy in a length dx, then from definition we have

$$p = E\frac{dy}{dx}.$$

The lamina dx will be urged forward—or backward—by the difference of the elastic forces on opposite sides of it, and as the quantities are infinitesimal, this difference will be dp; or

$$p' = dp = E\frac{d^2y}{dx}.$$

Let D be the density of the lamina, then its mass will be $M = Ddx$, and we have from equation (21), page 18,

$$Ddx\frac{d^2y}{dt^2} = E\frac{d^2y}{dx},$$

or,

$$\frac{d^2y}{dt^2} = \frac{E}{D}\frac{d^2y}{dx^2}, \tag{449}$$

which is a partial differential equation of the motion of any lamina. Let $E \div D = a^2$, and adding $a\dfrac{d^2y}{dxdt}$ to both members, we have

$$\frac{1}{dt}d\left(\frac{dy}{dt} + a\frac{dy}{dx}\right) = \frac{a}{dx}d\left(\frac{dy}{dt} + a\frac{dy}{dx}\right).$$

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tome II., has discussed the problem of the movement of a heavy liquid in a very long canal; M. Navier published a Mémoire on the flow of elastic fluids in pipes, in the *Académie des Sciences*, tome IX.; and M. Poisson wrote several Memoirs on the propagation of wave movements in an elastic medium, and the theory of sound, for which see *Journal de l'École Polytechnique*, 14th chapter, and of the *Académie of Sciences*, tomes II. and X. These eminent mathematicians established the basis of all the analysis for the solution of the problem. More recently we have M. Lamé's *Leçons sur l'Élasticité des Corps solides*, and Lord Rayleigh's *Treatise on Sound*, both of which are works of great merit.

Let

$$V = \frac{dy}{dt} + a\frac{dy}{dx}, \tag{450}$$

then

$$\left(\frac{dV}{dt}\right) = \frac{d^2y}{dt^2}, \tag{451}$$

where the parenthesis indicates a partial differential coefficient and

$$\left(\frac{dV}{dx}\right) = a\,\frac{d^2y}{dx^2}; \tag{452}$$

and equations (449), (451), (452), give

$$\left(\frac{dV}{dt}\right) = a\left(\frac{dV}{dx}\right).$$

The total differential of $V = f(x, t)$ is

$$dV = \left(\frac{dV}{dx}\right)dx + \left(\frac{dV}{dt}\right)dt;$$

by substituting (452),

$$= \left(\frac{dV}{dx}\right)(dx + adt)$$

$$= \left(\frac{dV}{dx}\right)d(x + at),$$

and integrating,

$$V = F(x + at) = \frac{dy}{dt} + a\frac{dy}{dx}, \tag{453}$$

where F is any arbitrary function.

Similarly, subtracting $a\dfrac{d^2y}{dxdt}$ from (449),

$$V' = f(x - at) = \frac{dy}{dt} - a\frac{dy}{dx}. \tag{454}$$

Adding and subtracting (453) and (454), and we have the respective equations

$$\frac{dy}{dt} = \tfrac{1}{2}F(x + at) + \tfrac{1}{2}f(x - at),$$

$$\frac{dy}{dx} = \frac{1}{2a}F(x + at) - \frac{1}{2a}f(x - at).$$

But

$$y = f(x, t);$$

$$\therefore\ dy = \left(\frac{dy}{dx}\right) dx + \left(\frac{dy}{dt}\right) dt,$$

and substituting from above, gives

$$dy = \frac{1}{2a}F(x + at)d(x + at) - \frac{1}{2a}f(x - at)d(x - at);$$

integrating,

$$y = \psi(x + at) + \varphi(x - at), \qquad (455)$$

where ψ and φ are any arbitrary functions whatever. Their character and initial values must be determined from the conditions of the problem. The equation represents a wave both from and towards the origin. If the wave be from the origin only, the function may be suppressed, and we have

$$y = \psi(x + at), \qquad (456)$$

and differentiating,

$$\left(\frac{dy}{dx}\right) = \psi'(x + at),$$

which is the rate of dilation (the expansion or contraction of a prism of the air), and

$$\left(\frac{dy}{dt}\right) = a \cdot \psi'(x + at);$$

which is the velocity of a particle, and dividing the latter by the former,

$$\frac{dx}{dt} = a, \tag{457}$$

which is the velocity of the wave; hence,

$$u = a = \sqrt{\frac{E}{D}}. \tag{458}$$

The elasticity of air equals its tension; hence if p be the pressure per square foot, w the weight of a cubic foot, and H the height of a homogeneous atmosphere, equation (424), then

$$u = \sqrt{\frac{gp}{w}} = \sqrt{gH}; \tag{459}$$

hence *the velocity of sound should equal the velocity of a body falling through a height equal to one-half the height of a uniform atmosphere.*

This principle is applicable also to the vibration of elastic cords, and it is found that

The velocity of vibration of an elastic cord equals the velocity of a body falling freely through a height equal to half the length of the same cord whose weight would equal the tension.

Similarly, in water neither too shallow nor too deep,

The velocity of waves on the sea equals the velocity of a body falling freely through a height equal to half the depth of the sea.

It has been assumed that E and D remain constant in wave motion; but it was long since known that the results given by equation (458) for gases did not agree with those found by experiment, and La Place showed that the elasticity was increased by the action of the wave due to compression. It is necessary, therefore, to consider that the expression is correct only for ultimate values; or

$$u = \sqrt{\frac{dE}{dD}}. \tag{460}$$

Since $E \propto w^{\gamma}$, $dE = \frac{\gamma E}{w} dw = \frac{g\gamma E}{w} dD$,

$$\therefore \; u = \sqrt{\frac{g\gamma E}{w}} = \sqrt{g\gamma H}. \qquad (461)$$

250. To find the value of γ, we have from equation (461),

$$\gamma = \frac{w}{gE} u^2 ; \qquad (462)$$

by means of which it may be found when the velocity of sound in a gas of given weight and tension is known. In this way the values of γ have been found for a variety of substances, a few of which are given in Article 246. It is considered as constant for any given gas, and nearly constant for all the more perfect gases.

251. Remarks. It is difficult to find the specific heat of a gas under constant volume by direct experiment, but by means of equations (462) and (442) it may be readily computed when the specific heat at constant pressure is known. In this way the values of c_v, given in Article 246, were determined.

If the temperature of air in its quiescent state is uniform, the tension will vary as the density, equation (416); hence $p \div w$ will be constant, and the velocity of sound in any gas will be the same at all temperatures or densities, bearing in mind that the factor γ appears on account of the condensation resulting from the transmission of the wave. But if the temperature varies on account of external causes, we have, equations (435) and (461),

$$u = \sqrt{g\gamma \frac{p_0}{w_0} \frac{\tau}{\tau_0}} ; \qquad (463)$$

in which the velocity is made to depend upon the absolute temperature. If $\tau_0 = 493.2$, $p_0 \div w_0 = 26,221$ feet, equation

(424), and $g = 32\frac{1}{6}$; the velocity in dry air at any temperature τ, will be

$$u = \sqrt{\frac{32\frac{1}{6} \times 1.408 \times 26{,}221}{493.2}\tau} = 1{,}089.6\sqrt{\frac{\tau}{\tau_0}} \text{ feet.} \quad (464)$$

Examples.

1. Required the velocity of sound in air at the temperature of 0° F.

2. Required the velocity of sound in air at 95° F.

3. If the weight of a cubic foot of hydrogen at the tension of the air, 14.7 lbs. and 32° F. be 0.005592, required the velocity of sound in it at 80° F., γ being 1.4.

4. Required the velocity of sound along a steel bar, the coefficient of elasticity, p, being 29,000,000 pounds per square inch, and $w = 486$ pounds.

$$u = \sqrt{\frac{40{,}000{,}000 \times 144 \times 32\frac{1}{6}}{486}} = 16{,}625 \text{ feet per second.}$$

The velocity of sound in solids depends much upon their homogeneity. Experiment shows that its velocity is from four to sixteen times as great as in air. The above equations are no more than approximately correct for solids.

252. Velocity of discharge of gases through orifices. If the density and temperature remained uniform during flow, the law of discharge would be the same as for liquids, and the same formula would apply, and we would have for the velocity, $v^2 = 2gh$, where h would be the reduced head, being the height of a prism of the gas of uniform density equal to the density at the orifice. If the temperature be uniform, the density will vary according to Mariotte's law, or $p = \frac{p_0}{\delta_0}\delta$. Let u be the velocity, then will the mass passing a transverse section of area unity in a unit of time be δu, and the difference in pressures on opposite sides of a section will

be dp, which will equal the momentum of the elementary mass, or

$$dp = -\delta u du; \tag{465}$$

in which the quantities have contrary signs, since the greater the velocity, the less the difference of the pressures. Dividing by p,

$$\frac{dp}{p} = -\frac{\delta_0}{p_0} u du;$$

integrating,

$$\log \frac{p}{p_1} = \frac{\delta_0}{2p_0}(u_1^2 - u^2), \tag{466}$$

where p is the tension of the gas in the reservoir, p_1 the external pressure, and if the gas flows into the atmosphere it will be 14.7 lbs., u_1 the velocity of exit, and u the initial velocity in the receiver. If the receiver be large, and the orifice small, p may be considered constant and u zero, in which case we have

$$u_1 = \sqrt{\frac{2p_0}{\delta_0} \log \frac{p}{p_1}}; \tag{467}$$

in which, if $p_1 = 0$, $u_1 = \infty$, or the velocity of a gas into a vacuum would be infinite according to this hypothesis.

If the temperature and density both vary, the effective head will vary. Letting z be positive downwards, we have, from equation (417),

$$z = \int \frac{dp}{w}.$$

Following the method of Joule and Thomson, and considering gravity as constant, w will be a function of p and γ; and substituting and reducing by means of equations (448), we have

$$\frac{v^2}{2g} = z = \int_{p_2}^{p_1} \frac{p_0^{\frac{1}{\gamma}}}{w_0} \cdot \frac{dp}{p^{\frac{1}{\gamma}}} = \frac{\gamma}{\gamma - 1} \cdot \frac{p_0}{w_0} \cdot \frac{\tau_1}{\tau_0}\left[1 - \frac{\tau_2}{\tau_1}\right]; \tag{468}$$

in which p_1 is the tension within the reservoir, and p_2 that just outside; then if $\tau_2 = 0$, the flow will be into a vacuum,

and according to (468) the velocity will be a maximum, the value of which will be

$$v = \sqrt{2g \cdot \frac{\gamma}{\gamma - 1} \cdot \frac{p_0}{w_0} \cdot \frac{\tau_1}{\tau_0}}, \tag{469}$$

which is $\sqrt{\frac{2}{\gamma - 1}}$ times the velocity of sound; and for air becomes

$$v = 2{,}413 \sqrt{\frac{\tau_1}{\tau_0}} \text{ feet per second};$$

and if the air be 32° F., then

$$v = 2{,}413 \text{ feet per second},$$

or about 2.2 times the velocity of sound in air.

253. THE VOLUME OF GAS flowing out per second will be

$$Q = sv, \tag{470}$$

where s is the area of the contracted section.

254. THE WEIGHT of gas flowing out per second will be, equations (448), (468), 470),

$$w_2 = \frac{w_0 \tau_0}{p_0} \cdot \frac{p_1}{\tau_1} \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}. \tag{471}$$

$$w_2 Q = sp_1 \sqrt{\left[\frac{2g\gamma}{\gamma - 1} \cdot \frac{w_0 \tau_0}{p_0 \tau_1} \left(1 - \frac{\tau_2}{\tau_1}\right)\right]} \left(\frac{\tau_2}{\tau_1}\right)^{\frac{1}{\gamma - 1}}, \tag{472}$$

which is a maximum for

$$\left(\frac{\tau_2}{\tau_1}\right)^{\frac{\gamma - 1}{\gamma}} = \frac{p_2}{p_1} = \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma}{\gamma - 1}} = \left(\frac{\delta_2}{\delta_1}\right)^{\gamma};$$

which for air becomes

$$\frac{\tau_2}{\tau_1} = 0.8306, \quad \frac{p_2}{p_1} = 0.527, \quad \frac{\delta_2}{\delta_1} = 0.6345.$$

APPENDIX I.

SOLUTIONS OF PROBLEMS.

PROBLEMS.

CONSTRAINED MOTION.

1. *A particle is placed at the extremity of the vertical minor axis of a smooth ellipse, and is just disturbed; show that if it quit the ellipse at the end of the latus rectum the eccentricity must satisfy the equation* $e^6 + 5e^4 + 3e^2 = 5$.

Let v be the velocity of the particle at the end of the latus rectum, then

$$v^2 = 2gh = 2g\left(b - \frac{b^2}{a}\right). \qquad (1)$$

Also in equation (148), p. 193, we have

$$\rho = a(1 - e^2)(1 + e^2)^{\frac{3}{2}},$$

$$X = 0, \quad Y = -mg.$$

From the equation to the ellipse,

$$\frac{dx}{ds} = \frac{1}{\sqrt{1 + e^2}} \text{ for } x = a\varepsilon,$$

hence,

$$v^2 = ga(1 - e^2)(1 + e^2)^{\frac{3}{2}} \frac{1}{\sqrt{1 + e^2}} = ga(1 - e^4). \qquad (2)$$

Equating (1) and (2),

$$2ga\left(\sqrt{1 - e^2} - (1 - e^2)\right) = ga(1 - e^4),$$

or,

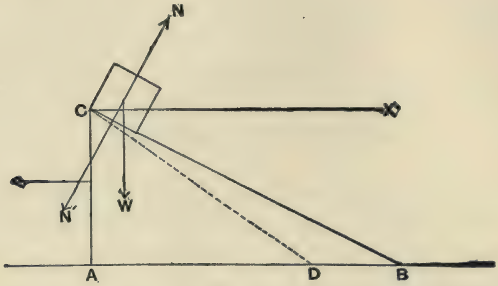
$$2\sqrt{1 - e^2} = (3 + e^2)(1 - e^2);$$

hence,

$$e^6 + 5e^4 + 3e^2 = 5.$$

2. *A particle slides down the upper surface of a frictionless*

wedge, the wedge being free to slide on a frictionless horizontal plane; determine the motions.



Let W = the weight of the body,
W' = the weight of the wedge,
$h = AC$, $b = AB$, $\theta = ACB$, N = normal action between the body and wedge,
R = the vertical reaction of the plane AB on which the wedge slides, $a = DB$ the distance moved by the wedge, $b - a = AD$, $ACD = \alpha$.

Take the origin of coördinates at C, x positive to the right, and y positive vertically down.

The forces acting on the body are N and W, and on the wedge N, W', and R. The angle θ_x, equations (143), is $-\theta$, and $\theta_y = 90° + \theta$; and the equations of motion become

$$\frac{W}{g}\frac{d^2x}{dt^2} = W\cos 90° + N\cos(-\theta) = N\cos\theta, \tag{1}$$

$$\frac{W}{g}\frac{d^2y}{dt^2} = W\sin 90° + N\sin(-\theta) = W - N\sin\theta; \tag{2}$$

$$\frac{W'}{g}\frac{d^2x'}{dt^2} = W'\cos 90° + N\cos(180 - \theta) + R\cos 270°$$

$$= -N\cos\theta. \tag{3}$$

$$\frac{W'}{g}\frac{d^2y'}{dt^2} = W'\sin 90° + N\sin(180 - \theta) + R\sin 270 = 0; \tag{4}$$

which is zero, since the plane is fixed, and hence there will be no motion in that direction.

From equations (1) and (3) we have

$$W\frac{d^2x}{dt^2} = -W'\frac{d^2x'}{dt^2};$$

integrating,

$$W\frac{dx}{dt} = -W'\frac{dx'}{dt};$$

and again,

$$Wx = -W'x';$$

or,

$$W(b-a) = W'a;$$

$$\therefore\ a = \frac{Wb}{W+W'}, \tag{5}$$

and

$$b - a = \frac{W'b}{W+W'}. \tag{6}$$

Let s be the distance from C down the plane, then

$$y = s\cos\theta,$$

$$x = s\sin\theta - \frac{a}{h}y;$$

differentiating,

$$d^2y = \cos\theta\, d^2s,$$

$$d^2x = \sin\theta\, d^2s - \frac{a}{h}d^2y,$$

reducing

$$= \left(\sin\theta - \frac{a}{h}\cos\theta\right)d^2s;$$

which, in (1) and (2), gives

$$\frac{d^2s}{dt^2} = \frac{gN\cos\theta}{W\left(\sin\theta - \frac{a}{h}\cos\theta\right)} = \frac{W - N\sin\theta}{W\cos\theta}g. \tag{7}$$

Solving,

$$N = \frac{WW'\sin\theta}{W'\sin^2\theta + (W+W')\cos^2\theta}, \tag{8}$$

and N is constant.

Integrating (1), (2), and (3), gives

$$W\frac{dx}{dt} = Wv_x = gN\cos\theta \cdot t, \tag{9}$$

$$W\frac{dy}{dt} = Wv_y = g\,(W - N\sin\theta)\,t, \tag{10}$$

$$W'\frac{dx'}{dt} = W'v_x' = -\,gN\cos\theta \cdot t\,; \tag{11}$$

and integrating again, gives

$$Wx \;= \tfrac{1}{2}gN\cos\theta \cdot t^2, \tag{12}$$

$$Wy \;= \tfrac{1}{2}g\,(W - N\sin\theta)\,t^2, \tag{13}$$

$$W'x' = -\,\tfrac{1}{2}gN\cos\theta \cdot t^2. \tag{14}$$

Making $x = b - a$ in (12), $x' = -\,a$ in (14), and we find equations (5) and (6) as before. Also make $y = h$ in (13), and find N as before. Then equations (8) and (13) give

$$t = \sqrt{\frac{2h}{g\cos^2\theta}\left(\cos^2\theta + \frac{W'}{W+W'}\sin^2\theta\right)} \tag{14'}$$

Integrating (1) and (2) in reference to x and y, and we have for the velocity with which the particle reaches the foot of the plane,

$$v = \sqrt{v_x^2 + v_y^2}$$

$$= \sqrt{2gh}\left[1 - \frac{WW'\sin^2\theta}{(W+W')[(W+W')\cos^2\theta + W'\sin^2\theta]^{\frac{1}{2}}}\right]^{\frac{1}{2}} \tag{15}$$

From (11) find

$$v_x' = -\sqrt{2gh}\left[\frac{WW'\sin\;\theta}{(W+W)(W+W')\cos^2\theta + W'\sin^2\theta}\right]^{\frac{1}{2}}. \tag{16}$$

The velocity with which the particle reaches the horizontal

plane is readily found by the principles of energy. Thus the work done by gravity equals Wh. Hence we have

$$Wh = \tfrac{1}{2}\frac{W}{g}v^2 + \tfrac{1}{2}\frac{W'}{g}v'^2; \tag{17}$$

which, reduced by the aid of (16), gives (15).

If $W' = \infty$, we have

$$N = W \sin\theta, \tag{18}$$

$$t = \sqrt{\frac{2h}{g\cos^2\theta}}, \tag{19}$$

$$v = \sqrt{2gh}, \tag{20}$$

$$v' = 0; \tag{21}$$

which are the formulas for the motion of a particle down a smooth *fixed* plane.

For the inclination of the path of the particle, we have

$$\tan\alpha = \frac{b-a}{h} = \frac{W'}{W+W'}\tan\theta.$$

A comparison of (15) and (19) shows that the particle acquires a less velocity when the wedge is free than when it is fixed.

QUESTIONS.—1. Show that whatever be the relative weights of the body and wedge, the common centre of gravity of the two masses will remain in the same vertical line during the motion down the wedge. (See Art. 149.)

2. Is the principle of the *conservation of energy*, Article 151, illustrated by this example?

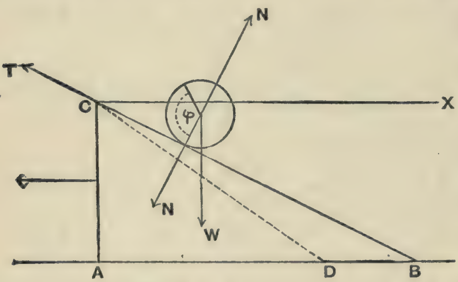
3. What is the relation between W and W', that will give a minimum velocity for the body W? (Eq. (15).)

4. Given W, equation (15), is it possible to assign such a value to W' or to θ, or to both, so that v will be $\tfrac{1}{2}\sqrt{2gh}$, $\tfrac{1}{4}\sqrt{2gh}$ or $\frac{1}{n}\sqrt{2gh}$?

5. Is there an algebraic minimum to the normal pressure, equation (8)—or an algebraic maximum?

3. *A homogeneous cylinder rolls down the upper surface of a wedge without slipping, the wedge being free to move on a frictionless horizontal plane; required the motion.*

The problem may be generalized, providing the successive elements of contact of the body and plane shall be in the successive elements of a circular cylinder circumscribing the body. The centre of gravity will move in a vertical plane, and the body may be a cylinder, a screw, a coil of wire, a sphere, an ellipsoid of revolution, a volume of revolution having two circles of contact with the centre of gravity between, and other forms. Rolling may be secured by a string (or flexible band) wound around the body, one end being secured at the upper end of the wedge, at C, and the other to the body. Let $W =$ weight of the body, k its radius of gyration, $W' =$ weight of the wedge, $N =$ normal reaction between the body and plane, T the tangential action producing rotation, $DB = z$, $AB = b$, $AD = b - z$, $AC = h$, $ACB = \theta$; origin at C; x positive to the right, and y positive downward, r the radius of the circle of contact.



The forces acting on the body are gravity, W, acting vertically downward, the normal reaction, N, acting directly away from the plane, and friction, or tension of the string T—whichever produces the rotation—acting upward along the plane.

For the equations of the motion of the centre of the body we have (167),

$$\frac{W}{g}\frac{d^2x}{dt^2} = W \cos 90° + N \cos(-\theta) + T \cos(270° - \theta), \quad (1)$$

$$\frac{W}{g}\frac{d^2y}{dt^2} = W \sin 90° + N \sin(-\theta) + T \sin(270° - \theta). \quad (2)$$

The forces acting upon the wedge are its weight, W', acting vertically downward, the normal reaction, N', of the horizontal plane acting vertically upward, the normal reaction, N, between the body and wedge acting downward, and the pull, T, acting along the upper surface of the plane downward. Hence we have

$$\frac{W'}{g}\frac{d^2x'}{dt^2} = W' \cos 90° + N' \cos 270° + N \cos (180° - \theta) + T \cos (90° - \theta). \quad (3)$$

The origin of moments being taken at the centre of the body, we have, equation (152), for the moment of rotation of the body,

$$Mk^2 \frac{d^2\varphi}{dt^2} = Tr. \quad (4)$$

These equations may be reduced to

$$\left.\begin{aligned} \frac{W}{g}\frac{d^2x}{dt^2} &= N \cos \theta - T \sin \theta \\ \frac{W}{g}\frac{d^2y}{dt^2} &= W - N \sin \theta - T \cos \theta \\ \frac{W'}{g}\frac{d^2x'}{dt^2} &= -N \cos \theta + T \sin \theta \\ \frac{W}{g}k^2\frac{d^2\varphi}{dt^2} &= Tr. \end{aligned}\right\} \quad (5).$$

Integrating twice, assuming N and T constant, as they are,

$$\frac{W}{g}x\Big]_0^{b-z} = \frac{W}{g}(b - z) = \tfrac{1}{2}(N \cos \theta - T \sin \theta)\, t^2, \quad (6)$$

$$\frac{W}{g}y\Big]_0^{h} = \frac{W}{g}h = \tfrac{1}{2}(W - N \sin \theta - T \cos \theta)\, t^2, \quad (7)$$

$$\frac{W'}{g}x'\Big]_z^{0} = -\frac{W'}{g}z = \tfrac{1}{2}(-N \cos \theta + T \sin \theta)\, t^2, \quad (8)$$

$$\frac{W}{g}k^2\varphi = \tfrac{1}{2}Trt^2. \quad (10)$$

When the body rolls completely down the wedge, we have

$$r\varphi = CB = \sqrt{h^2 + b^2} = a \text{ (say)} \tag{11}$$

which in the preceding equation gives

$$T = \frac{2Wk^2a}{r^2t^2g}, \tag{12}$$

which reduces the preceding equations to

$$W(b - z) = \tfrac{1}{2}gN\cos\theta \cdot t^2 - \frac{W\,k^2a\sin\theta}{r^2}, \tag{13}$$

$$Wh = \tfrac{1}{2}g\,(W - N\sin\theta)\,t^2 - \frac{Wk^2a\cos\theta}{r^2} \tag{14}$$

$$W'z = \tfrac{1}{2}gN\cos\theta \cdot t^2 - \frac{Wk^2a\sin\theta}{r^2}. \tag{15}$$

Subtracting (15) from (13), gives

$$z = \frac{Wb}{W + W'}, \tag{16}$$

$$\therefore\ b - z = \frac{W'b}{W + W'}. \tag{17}$$

From (13) and (17),

$$t^2 = \left[\frac{WW'b}{W + W'} + \frac{Wk^2a\sin\theta}{r^2}\right]\frac{2}{g\cos\theta\,N} = \frac{A}{N} \text{ (say)}$$

from (14),

$$t^2 = \left[Wh + \frac{Wk^2a\cos\theta}{r^2}\right]\frac{2}{g\,(W - N\sin\theta)};$$

$$= \frac{B}{W - N\sin\theta} \text{ (say)};$$

$$\therefore\ N = \frac{AW}{B + A\sin\theta}$$

$$t = \sqrt{\frac{B + A \sin \theta}{W}}$$

$$T = \frac{2W^2k^2a}{r^2g(B + A \sin \theta)} = \frac{Wk^2 \cos \theta}{r^2 \sin^2 \theta \dfrac{W'}{W + W'} + r^2 \cos^2 \theta + k^2}$$

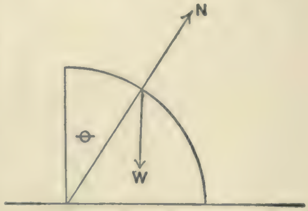
$$v = \left[\frac{A^2r^4g^2\cos^2\theta + B^2r^4g^2 - 4Wk^2ar^2g\cos\theta(B + A\sin\theta) + 4W^2k^4a^2}{r^4W(B + A \sin \theta)}\right]^{\frac{1}{2}}$$

$$v_x = \sqrt{\frac{W}{B + A \sin \theta}}\left(\frac{2Wk^2a - r^2gA}{r^2W'}\right).$$

The entire work done by gravity is Wh, and this equals the energy of the body due to the velocity of the centre, plus the energy due to the rotation of the body, plus the energy due to the motion of the wedge; or

$$Wh = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}Mk^2\omega^2 + \tfrac{1}{2}M'v'^2.$$

4. *A particle slides down a frictionless arc of a fixed vertical circle; required the motion.*



The forces acting on the particle will be gravity, W, and the normal action, N, between the body and arc acting normally outward, and will be the difference between the normal component of the weight and centrifugal force. Using polar coordinates, origin at the centre, θ the angle between r, the radius of the arc, and a vertical diameter, and resolving normally and tangentially, we have

$$\frac{W}{g} r \frac{d^2\theta}{dt^2} = W \sin \theta, \qquad (1)$$

$$\frac{W}{g} r \left(\frac{d\theta}{dt}\right)^2 = W \cos \theta - N. \qquad (2)$$

Integrating (1) gives

$$r\frac{d\theta^2}{dt^2} = -2g\cos\theta\Big]_0^\theta = 2g(1-\cos\theta), \tag{3}$$

which is independent of the mass. In (2) this gives

$$N = (3\cos\theta - 2)W. \tag{4}$$

At the point where the body leaves the arc $N = 0$, for which condition (4) gives

$$\cos\theta = \tfrac{2}{3}; \tag{5}$$

$$\therefore\ \theta = 48^\circ\ 11' +.$$

The velocity at this point is, equation (3),

$$r\frac{d\theta}{dt} = \sqrt{\tfrac{2}{3}gr}. \tag{6}$$

The subsequent motion will be that of a projectile, having the initial velocity of (6) and angle of depression of $48^\circ\ 11' +$.

The normal pressure, N, varies inversely as θ, as shown by (4).

To find the time of the contact, we have from (3),

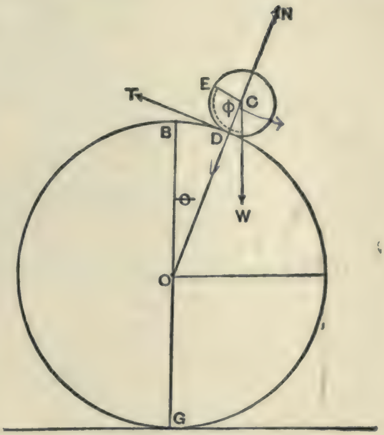
$$t = \sqrt{\frac{r}{2g}}\int_{\cos^{-1}\frac{2}{3}}^{0}\frac{d\theta}{\sqrt{1-\cos\theta}}$$

$$= \sqrt{\frac{r}{g}}\log\tan\tfrac{1}{4}\theta\Big]_{\cos^{-1}\frac{2}{3}}^{0}.$$

The superior limit makes $t = -\infty$, which shows that the particle will not start from rest at the highest point. It will be necessary either to give it an initial velocity, or place it at a finite distance from the highest point.

5. *A body rolls down the arc of a fixed circle without sliding, starting at a point indefinitely near the highest point of the arc; required the motion.*

Let O be the centre of the arc GB, C the centre of the rolling body, $W =$ weight of the body, $R = OD$, $r = DC$, the radius of the circle of contact, $k =$ the principal radius of gyration in reference to a horizontal axis through C, $\theta = BOD$. Resolving tangentially and normally, we have for the motion of the centre C,



$$\frac{W}{g}(R + r)\frac{d^2\theta}{dt^2} = W \sin \theta + N \sin 180° + T \sin 270°,$$

$$\frac{W}{g}(R + r)\left(\frac{d\theta}{dt}\right) = W \cos \theta + N \cos 180° + T \cos 270°,$$

or

$$W(R + r)\frac{d^2\theta}{dt^2} = Wg \sin \theta - Tg, \tag{1}$$

$$W(R + r)\left(\frac{d\theta}{dt}\right)^2 = Wg \cos \theta - Ng. \tag{2}$$

And for rotation we have (152),

$$\frac{W}{g}k^2\frac{d^2\varphi}{dt^2} = Tr; \tag{3}$$

where φ is the angle which some fixed line CE of the body makes with the vertical; thus φ will be zero when θ is zero. Since arc $ED =$ arc BD, and $DCW = \theta$, $DCE = \varphi - \theta$, we have

$$r(\varphi - \theta) = R\theta;$$

$$\therefore \varphi = \frac{R + r}{r}\theta. \tag{4}$$

Solving these equations gives

$$T = \frac{k^2}{k^2 + r^2} W \sin \theta. \tag{5}$$

$$N = \left[3 \cos \theta + \frac{2k^2}{k^2 + r^2}(1 - \cos \theta) - 2\right] W. \qquad (6)$$

When equation (6) becomes zero the body leaves the arc. Making $N = 0$, we find

$$\cos \theta = \frac{2r^2}{k^2 + 3r^2}. \qquad (7)$$

If the body be a sphere we have $k^2 = \frac{2}{5}r^2$, and

$$\cos \theta = \tfrac{10}{17};$$

$$\therefore \ \theta = 53^\circ \ 58'.$$

To find the velocity of the centre of the sphere when it leaves the arc, substitute T from (5) in (1) and we find

$$(R + r)\frac{d^2\theta}{dt^2} = \tfrac{5}{7}g \sin \theta, \qquad (8)$$

and integrating and reducing, we find

$$(R + r)\frac{d\theta}{dt} = \sqrt{\tfrac{10}{7}g(1 - \cos \theta)(R + r)}, \qquad (9)$$

which is the velocity at any distance θ from the vertical. Making $\cos \theta = \frac{10}{17}$, we have for the required velocity,

$$\sqrt{\tfrac{10}{17}g(R + r)}. \qquad (10)$$

To find the angular velocity of the sphere at the point where it leaves the arc, substitute $d\theta$ found from (4) in (9) and make $\cos \theta = \frac{10}{17}$; we thus find

$$\omega = \sqrt{\frac{10g}{17r^2}(R + r)}. \qquad (11)$$

From this point the sphere will continue with the uniform angular velocity given in (11), (the body having rolled by friction, or at that point being freed from the string), and with the initial velocity given by (10), the centre moving as a projectile. The sphere will strike the horizontal plane tangent at the lowest

point of the complete circular arc, when its centre is at a distance r above that plane; hence the sphere will be a projectile through the height

$$h = R \cos \cos^{-1} \tfrac{10}{17} + R - r$$

$$= \tfrac{27}{17} R - r. \qquad (12)$$

The time of movement as a projectile will be given by equations (a), p. 177, and will be given by the equation

$$t^2 + 0.2254 \sqrt{R + r} \cdot t = 0.10492 R - 0.062177 r.$$

During contact it will rotate on its axis

$$\frac{R}{2\pi r} \cos^{-1} \frac{10}{17}$$

times, and as a projectile it will rotate

$$\frac{\omega t}{2\pi} \text{ times.}$$

If the arc be not infinitely rough, let μ be the coefficient of friction, rotation being caused entirely by friction; *required the motion for this condition.*

The friction will be μN, and so long as $T < \mu N$ the body will roll, but from the point where $T = \mu N$ the body will slide on the arc, and the force producing rotation will be μN. The first part of the motion will be given by equations (1), (2), and (3); but during the latter part μN must be substituted for T in (1) and (3), and the equations integrated again, the inferior limits for θ and φ being the values found from (4), (5), (6), and $T = \mu N$. During this part of the motion, equation (4) will not be true, but instead thereof, if $s = r\varphi'$ be the total arc *slipped* over, we have

$$R\theta = r(\varphi - \theta + \varphi'). \qquad (13)$$

That there be no slipping we must have

$$T < \mu N,$$

or from equations (5) and (6),

$$\frac{k^2}{k^2 + r^2} \sin \theta < \mu \left(3 \cos \theta + \frac{2k^2}{k^2 + r^2}(1 - \cos \theta) - 2 \right). \qquad (14)$$

Equation (3) becomes

$$\frac{W}{g} k^2 \frac{d^2\varphi}{dt^2} = \mu Nr, \tag{15}$$

and (1),

$$W(R + r) \frac{d^2\theta}{dt^2} = Wg \sin \theta - \mu Ng, \tag{16}$$

and (2) remains,

$$W(R + r) \left(\frac{d\theta}{dt}\right)^2 = Wg \cos \theta - Ng. \tag{17}$$

Eliminating N between (16) and (17) gives

$$W(R + r) \left[\frac{d^2\theta}{dt^2} - \mu \left(\frac{d\theta}{dt}\right)^2\right] = Wg (\sin \theta - \mu \cos \theta),$$

which, if it could be integrated, would give $\theta = f(t)$, and then $\frac{d\theta}{dt}$ in (17) would give N, which in (15) makes known, by integration, $\varphi = F(t)$; and these results combined with (13) would make known φ' in terms of t. The time of the movement and the superior limit of θ during contact would be found by making $N = 0$ in the value found for N. If the body had a rolling motion only, it would leave the arc when $\theta = 53° \ 58'$, as shown by (7) above, and if it slid off without friction, starting from the highest point, it would leave it at $\theta = 48° \ 11'$, as shown by the preceding problem. If the body both slips and rolls, it will leave it at a point whose angular distance from the vertical is less than 54°, and greater than 48°.

6. What must be the radius of the rolling body so that it will touch the horizontal plane at the instant it leaves the arc, after having rolled from the highest point?

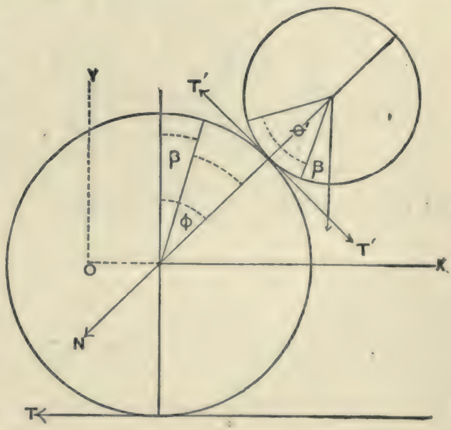
7. What is the relation between the radius of the body and of the arc that the body shall roll twice on the arc from the highest point, before leaving it?

8. What must be the relation between the radius of the arc and that of the sphere, so that by rolling from the highest point of the arc to the horizontal plane through the lowest point of the circular arc the body will rotate once, or twice, or n times?

9. The base of a hemisphere rests on a horizontal plane, and a sphere rests at its highest point; if, from a slight disturbance, the sphere *rolls* off the hemisphere, how far from the centre of the hemisphere will it strike the horizontal plane, and what will be the angular distance from the point where the sphere strikes the plane to the point on the sphere originally in contact with the hemisphere.—(Solution in *The Mathematical Visitor*, Jan., 1881, p. 191.)

10. *One sphere rolls down another, the latter being free to roll on a horizontal plane; required the motions.*

Let R be the radius of the lower sphere, M its mass, θ the angle through which it will have rolled in time t from a fixed vertical line; r, m, θ', corresponding values for the upper sphere, φ the angle between the line of the centres and the vertical, T' the tangential action between the spheres, T the friction on the horizontal plane, and N the normal action between the spheres.



Take the origin of coordinates at the centre of the lower sphere, x horizontal and positive to the right, y vertically upward, and β the initial angle of φ. For the sake of distinction, let x' and y' apply to the upper sphere. The figure represents the condition of the bodies at the end of time t.

The tangential action between the spheres will, at first, act downward in reference to the lower sphere and upward in reference to the upper sphere. The normal action, N, will not produce rolling of the upper sphere, but will tend to produce rolling of the lower one in an opposite direction to that of the tangential action. Any modification of these assumptions due to the motion will appear in the solution.

For the lower sphere we have

$$\left.\begin{aligned}
M\frac{d^2x}{dt^2} &= N\cos(270^\circ+\varphi) + T'\cos(360^\circ-\varphi) + T\cos 180^\circ \\
&= -N\sin\varphi + T'\cos\varphi - T \\
M\frac{d^2y}{dt^2} &= 0 \\
Mk^2\frac{d^2\theta}{dt^2} &= TR + T'R \\
\text{or } \tfrac{2}{5}MR\frac{d^2\theta}{dt^2} &= T + T'
\end{aligned}\right\};(1)$$

and for the upper sphere,

$$\left.\begin{aligned}
m\frac{d^2x'}{dt^2} &= N\sin\varphi - T'\cos\varphi \\
m\frac{d^2y'}{dt^2} &= N\cos\varphi + T'\sin\varphi - mg \\
\tfrac{2}{5}mr\frac{d^2\theta'}{dt^2} &= T'
\end{aligned}\right\}; \qquad (2)$$

and for the geometrical conditions,

$$x = R\theta; \qquad dx = Rd\theta, \qquad (3)$$

$$x' - x = (R+r)\sin\varphi; \;\therefore\; dx' - dx = (R+r)\cos\varphi d\varphi, \quad (4)$$

$$y' = (R+r)\cos\varphi; \;\therefore\; dy' = -(R+r)\sin\varphi d\varphi, \qquad (5)$$

$$R(\varphi-\theta-\beta) = r(\theta'-\varphi+\beta); \; R(d\varphi-d\theta) = r(d\theta'-d\varphi). \qquad (6)$$

Eliminating N, T, and T' from equations (1) and (2), gives

$$\tfrac{2}{5}MR\frac{d^2\theta}{dt^2} + M\frac{d^2x}{dt^2} + m\frac{d^2x'}{dt^2} = \tfrac{2}{5}mr\frac{d^2\theta'}{dt^2}. \qquad (7)$$

Integrating once, the initial values being 0, gives

$$M\frac{dx}{dt} + m\frac{dx'}{dt} = \tfrac{2}{5}mr\frac{d\theta'}{dt} - \tfrac{2}{5}MR\frac{d\theta}{dt}; \qquad (8)$$

from which we see that *when* $\dfrac{d\theta}{dt} = \dfrac{mr}{MR}\cdot\dfrac{d\theta'}{dt}$, *we have* $MV = -mv$; *or the moments are numerically equal, and the centres*

are moving in opposite directions, (V and v being respectively the velocities of the centres).

Eliminating $d\theta$, $d\theta'$, and dx' by means of (3), (4), and (6), gives

$$\tfrac{7}{5}(M+m)\frac{dx}{dt} = m(R+r)(\tfrac{2}{5} - \cos\varphi)\frac{d\varphi}{dt}, \qquad (9)$$

and

$$\tfrac{7}{5}(M+m)\frac{dx'}{dt} = [m(\tfrac{2}{5} - \cos\varphi) + \tfrac{7}{5}(M+m)\cos\varphi](R+r)\frac{d\varphi}{dt}. \quad (10)$$

From (9) it appears that *the motion of the centre of the lower sphere will be positive so long as* $\cos\varphi$ *is less than* $\frac{2}{5}$, or $\varphi < 66°\ 26'$. Should the spheres separate at a less angle, they will continue to roll in the same direction.

From (9) it appears that when $\cos\theta = \frac{2}{5}$, the centre of the lower sphere will be at rest, in which case (3) shows that there will be no rotary motion; in short, the lower sphere will then be at rest. Equation (10) shows that the motion of the upper sphere will be continually positive. By means of the equations given above, a complete solution may be found, giving the normal reactions and angular velocities.

11. *A uniform rod of length* a, *capable of making complete revolutions in a vertical plane about one extremity, is placed in a vertical position with its free end upward, and being slightly displaced, moves from rest; find the time of revolving from an angle* β *to an angle* θ.

The equation of motion is, Art. 126,

$$m\frac{a^2}{3}\frac{d^2\theta}{dt^2} = \frac{a}{2}mg\sin\theta,$$

or

$$\frac{2}{3}\frac{a}{g}\frac{d^2\theta}{dt^2} = \sin\theta.$$

Multiplying both members by $2d\theta$ and integrating between the limits θ and 0, we have

$$\frac{2a}{3g}\left(\frac{d\theta}{dt}\right)^2 = 2(1 - \cos\theta),$$

or

$$\frac{dt}{d\theta} = \frac{1}{2}\left(\frac{2a}{3g}\right)^{\frac{1}{2}} \operatorname{cosec} \tfrac{1}{2}\theta;$$

$$\therefore t + C = \left(\frac{2a}{3g}\right)^{\frac{1}{2}} \log \tan \tfrac{1}{4}\theta.$$

Hence the time from β to θ is

$$t = \left(\frac{2a}{3g}\right)^{\frac{1}{2}} \log \frac{\tan \frac{1}{4}\theta}{\tan \frac{1}{4}\beta}.$$

12. *A rod rests with one extremity on a smooth plane and the other against a smooth vertical wall at an inclination α to the horizon. If it then slips down, show that it will leave the wall when its inclination is $\sin^{-1}(\frac{2}{3}\sin\alpha)$.*

Let the mass of the rod be m, its length $2l$, its inclination to the horizon θ, and the coördinates of its centre of gravity x and y; the origin being such that for the time, t, considered, $x = l\cos\theta$ and $y = l\sin\theta$. Let the horizontal and vertical reactions at the ends of the rod be H and V respectively. Then the equations of motion are

$$m\frac{d^2y}{dt^2} = ml\frac{d^2(\sin\theta)}{dt^2} = -mg + V, \tag{1}$$

$$m\frac{d^2x}{dt^2} = ml\frac{d^2(\cos\theta)}{dt^2} = H, \tag{2}$$

$$\tfrac{1}{3}ml^2\frac{d^2\theta}{dt^2} = Hl\sin\theta - Vl\cos\theta. \tag{3}$$

Multiply (1) by $\cos\theta$, (2) by $-\sin\theta$, and (3) by $(1 \div l)$, and add the products. There results

$$\tfrac{4}{3}ml\frac{d^2\theta}{dt^2} = -mg\cos\theta, \tag{4}$$

whence, since $\frac{d\theta}{dt} = 0$ and $\theta = \alpha$ when $t = 0$,

$$\frac{d\theta^2}{dt^2} = \frac{3}{2}\frac{g}{l}(\sin\alpha - \sin\theta). \tag{5}$$

The rod will leave the vertical wall when

$$H = -ml\left(\sin\theta\frac{d^2\theta}{dt^2} + \cos\theta\frac{d\theta^2}{dt^2}\right) = 0.$$

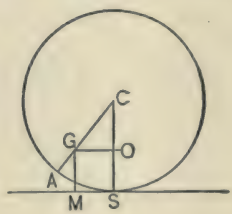
Substituting in this the values of $\frac{d^2\theta}{dt^2}$ and $\frac{d\theta^2}{dt^2}$ given by (4) and (5) we have

$$\theta = \sin^{-1}(\tfrac{2}{3}\sin\alpha).$$

(*The Analyst*, 1882, p. 193.)

13. *An angular velocity having been impressed on a heterogeneous sphere, about an axis perpendicular to the vertical plane which contains its centre of gravity G, and geometrical centre C, and passing through G, it is then placed on a smooth horizontal plane. Find the magnitude of the impressed angular velocity that G may rise into a point in the vertical line SCK through C, and there rest; the angle GCS being α at the beginning of the motion, a the radius, and φ the required angular velocity.*

Draw the radius CGA, and from G drop the perpendicular GM to the plane.



Let m = the mass of the sphere, k the radius of gyration of the sphere about an axis through G perpendicular to the plane containing C and G, R the mutual reaction of the sphere and the plane, $SM = x$, $GM = y$, $CS = a$, angle AGM = angle $ACS = \varphi$, and $CG = c$.

Since there is no friction, we have for the motion of the centre of the sphere

$$m\frac{d^2x}{dt^2} = 0, \tag{1}$$

resolving forces vertically,

$$m\frac{d^2y}{dt^2} = R - mg, \tag{2}$$

and taking moments about G,

$$mk^2\frac{d^2\varphi}{dt^2} = -Rc\sin\varphi, \tag{3}$$

φ being the angular motion of the sphere.

We have $y = a - c\cos\varphi$, whence

$$\frac{d^2y}{dt^2} = c\sin\varphi\frac{d^2\varphi}{dt^2} + c\cos\varphi\frac{d\varphi^2}{dt^2}.$$

Substituting in (2),

$$R = m\left(c\sin\varphi\frac{d^2\varphi}{dt^2} + c\cos\varphi\frac{d\varphi^2}{dt^2} + g\right).$$

This in (3) gives by reduction,

$$(c^2\sin^2\varphi + k^2)\frac{d^2\varphi}{dt^2} + c^2\sin\varphi\cos\varphi\frac{d\varphi^2}{dt^2} = -cg\sin\varphi. \tag{4}$$

Integrating,

$$(c^2\sin^2\varphi + k^2)\frac{d\varphi^2}{dt^2} = C + 2cg\cos\varphi. \tag{5}$$

Let $t = 0$, when $\varphi = \alpha$; $\dfrac{d\varphi}{dt} = \omega$, and

$$C = (c^2\sin^2\alpha + k^2)\omega^2 - 2cg\cos\alpha.$$

Hence (5) becomes

$$(c^2\sin^2\varphi + k^2)\frac{d\varphi^2}{dt^2} = (c^2\sin^2\alpha + k^2)\omega^2 + 2cg(\cos\varphi - \cos\alpha). \tag{6}$$

Now if the initial value of $\varphi = \alpha$, the terminal value $= \pi - \alpha$, when also $\dfrac{d\varphi}{dt} = 0$; then the left member of (6) becomes zero, and

$$\omega^2 = \frac{4cg\cos\alpha}{c^2\sin^2\alpha + k^2}.$$

(*The Analyst*, July, 1882.)

Kinetic Energy.

14. ***A ball, mass m, radius r, is shot with a velocity v into a perfectly hard, smooth tube of length a, radius r', mass m', free to turn about its middle point, which is fixed, imparting to the tube a rotary motion; if the ball just reaches the centre of the tube, required the angular velocity of the latter.***

In this case the kinetic energy of the ball and tube due to the rotary motion, will equal the kinetic energy of the ball before it enters the tube. Let k and k' be the radii of gyration of the ball and tube respectively, and ω the required angular velocity, then, equation (153),

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mk^2\omega^2 + \tfrac{1}{2}m'k'^2\omega^2;$$

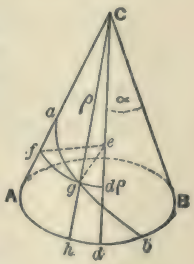
$$\therefore \omega^2 = \frac{mv^2}{mk^2 + m'k'^2}.$$

If the tube be considered slender, we will have $k'^2 = \frac{1}{12}a^2$; also $k^2 = \frac{2}{5}r^2$, hence,

$$\omega^2 = \frac{60mv^2}{24mr^2 + 5m'a^2}.$$

15. *A cone, mass m and vertical angle 2α, is perfectly free to move about its axis, and has a fine, perfectly smooth groove cut in its surface, making constant angle β with the elements of the cone. A heavy particle, mass M, moves along the groove under the action of gravity, starting at a distance c from the vertex; required the angle through which the cone has turned when the particle is at a distance r from the vertex.*

Let ρ be any variable distance from the vertex, φ the angle through which the particle has moved, k the radius of gyration of the cone, and θ the required angle.



The geometrical relations give

$\rho \sin \alpha$ = radius of the cone at the place of the particle at time t,

$\rho \sin \alpha d\varphi$ = the horizontal arc through which the particle moves in time dt,

$d\rho \tan \beta$ = the same arc;

$$\therefore\ d\rho = \rho \sin\alpha \cot\beta d\varphi. \qquad (1)$$

The moment of the momentum imparted to the cone in an element of time, will be

$$mk^2 \frac{d\theta}{dt}.$$

The angular advance of the particle will be $d\varphi - d\theta$, and the horizontal component of the momentum of the particle will be

$$M\rho \sin\alpha \left(\frac{d\varphi - d\theta}{dt}\right);$$

and the moment of the momentum will be

$$M\rho^2 \sin^2\alpha \left(\frac{d\varphi - d\theta}{dt}\right).$$

Since gravity has no horizontal component, the horizontal motions will be due to the action and reaction between the bodies, and these moments must be equal ; hence

$$M\rho^2 \sin^2\alpha \left(\frac{d\varphi - d\theta}{dt}\right) = mk^2 \frac{d\theta}{dt}. \qquad (2)$$

Eliminating $d\varphi$ by means of (1) and (2), we have

$$\int_c^r \frac{2M \sin^2\alpha \rho d\rho}{mk^2 + M\rho^2 \sin^2\alpha} = \int_0^\theta 2 \sin\alpha \cot\beta d\theta;$$

$$\therefore\ \theta = \tfrac{1}{2} \frac{\tan\beta}{\sin\alpha} \log \frac{mk^2 + Mr^2 \sin^2\alpha}{mk^2 + Mc^2 \sin^2\alpha}.$$

16. An elastic ring, mass m, natural radius a, modulus of elasticity ε, is stretched around a cylinder; the cylinder suddenly vanishes; find the time in which the ring will collapse to its natural length.

$$T = \sqrt{\frac{\pi a m}{8\varepsilon}}.$$

17. *A prismatic bar in a vertical position rests on a pivot at its lower extremity ; it is slightly disturbed, required its kinetic energy when it will have rotated* 180°.

We have, Prob. 19, p. 216, and Art. 107,

$$-\frac{d^2\theta}{dt^2} = g\frac{W\cdot\frac{1}{2}l\sin\theta}{W\cdot\frac{1}{3}l^2} = \frac{3}{2}\frac{g}{l}\sin\theta;$$

and integrating

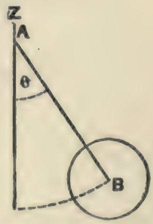
$$\frac{1}{2}\frac{d\theta^2}{dt^2} = \frac{3}{2}\frac{g}{l}\cos\theta\Big]_\pi^0 = 3\frac{g}{l};$$

which in equation (153) gives

$$\tfrac{1}{2}\cdot\tfrac{1}{3}ml^2\cdot\frac{d\theta^2}{dt^2} = Wl;$$

that is, *the energy is the same as if the bar had fallen freely through the vertical descent of the centre of gravity of the bar.*

18. *If a sphere, pivoted on a horizontal diameter as an axis without friction, oscillates about an external axis; required the kinetic energy when vertically under the support.*



Let the body rotate about the axis of y, then will equations (177) be applicable, and we shall have

$AB = l$, $L_1 = 0$, $M_1 = -Wl\sin\theta$, $N_1 = 0$, $Mg = W$, $x = l\sin\theta$, $z = l\cos\theta$;

$$\therefore\ d^2x = -l\sin\theta d\theta^2 + l\cos\theta d^2\theta,$$
$$d^2z = -l\cos\theta d\theta^2 - l\sin\theta d^2\theta;$$

and these, in equations (177), give

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta;$$

which, integrated, gives

$$\frac{1}{2}\frac{d\theta^2}{dt^2} = \frac{g}{l}\cos\theta\Big]_\theta^0 = \frac{g}{l}(1-\cos\theta).$$

The velocity of the centre of the sphere at the lowest point will be

$$l\frac{d\theta}{dt} = \sqrt{2gl(1 - \cos\theta)},$$

and the kinetic energy of the mass when it rotates through 180° will be

$$\tfrac{1}{2}M \cdot l^2 \frac{d\theta^2}{dt^2} = 2Mgl = 2Wl;$$

hence in this case also, the kinetic energy is the same as if the sphere had fallen freely through the height equal to the descent of the centre of gravity.

19. *Suppose, in the preceding example, that the axis of the sphere be rigidly connected with the rod AB; required the velocity.*

Here we have, as in example 19, p. 216,

$$-\frac{d^2\theta}{dt^2} = \frac{gl}{l^2 + k_1^2}\sin\theta;$$

$$\therefore \frac{1}{2}\frac{d\theta^2}{dt^2} = \frac{gl}{l^2 + \frac{2}{5}r^2}(1 - \cos\theta);$$

which, compared with the preceding problem, shows that *the angular velocity will be less at the lowest point when the sphere is rigidly connected with the rod AB, than when it is free to roll on its own axis.*

The kinetic energy in this case, when it will have rotated from the highest to the lowest point, will be, equation (153), page 202,

$$\tfrac{1}{2} \cdot m(l^2 + \tfrac{2}{5}r^2) \cdot \frac{d\theta^2}{dt^2} = 2Wl;$$

which is the same as in the preceding case. The time of vibration will be greater in this case than in the preceding—equation (161).

QUERIES.—1. If the sphere in example 18 be free to rotate on its horizontal axis as a diameter while the entire mass rotates about an external axis, and it rotates through 180°, starting with no velocity from a point vertically over the fixed axis; if, at the lowest point in its path its axis instantly becomes rigid with the bar AB, will it rise to the highest point?

2. In the preceding example, will the time of describing the second part of the arc be the same as that of describing the first 180° ?

3. In example 18, if the sphere gradually melts away, will the velocity or time of vibration be thereby affected ?

4. In example 19, if the sphere gradually melts away, will the time or velocity be thereby affected ?

5. In examples 18 and 19, which will produce the greater stress on the axis of suspension, the masses and arcs of vibration being the same in both cases ?

6. At what points of the rotating masses must they strike a fixed obstacle so as to produce no shock on the fixed axis ?

7. If a spherical shell B be rigidly connected to the bar AB and filled with a frictionless fluid, would the time of vibration and the velocity at the lowest point remain the same if the fluid should suddenly freeze ?

8. If a spherical shell rigidly connected to a bar and filled with a frictionless fluid, be rotating about a vertical axis with a uniform velocity under the action of no forces ; should the fluid suddenly freeze, will the velocity of rotation remain the same ? Will the kinetic energy remain the same ?

Conservation of Areas (Art. 150). Moment of the Momentum (Art. 166).

20. *A cylinder of ice, radius r, length l, revolves with a uniform angular velocity ω ; if it be subject to no external force and melts, required the angular velocity of the resulting sphere.*

Neglecting the contraction due to melting, and the spheroidal form due to rotation, and letting $\omega' =$ the required angular velocity, $k^2 = \frac{2}{5}R^2 =$ the principal radius of gyration, we have, Article 166,

$$m \cdot \tfrac{1}{2}r^2\omega = m \cdot \tfrac{2}{5}R^2\omega',$$

$$\therefore\ \omega' = \frac{5}{4}\frac{r^2}{R^2}\omega.$$

To find R we have

$$\tfrac{4}{3}\pi R^3 = \pi r^2 l;$$

$$\therefore\ R^2 = (\tfrac{3}{4}r^2 l)^{\frac{2}{3}};$$

$$\therefore\ \omega' = \frac{5}{4}\left(\frac{4}{3}\frac{r}{l}\right)^{\frac{2}{3}}.$$

21. *If n spherical shells of infinitesimal thickness, mass m of each, radius r, move in contact without friction, having their axes of rotation all in one plane, the angles between the axes and an assumed line being $\beta_1, \beta_2, \beta_3, \ldots \beta_n$, and having angular velocities $\omega_1, \omega_2, \omega_3, \ldots \omega_n$, respectively, suddenly become solid; required the position of the resultant axis, and the resultant angular velocity.*

Let θ be the required angle and ω the required angular velocity; $k^2 = \frac{2}{3}r^2$, the principal radius of gyration before and after becoming solid; then, since the moment of the momentum will be constant, we have, resolving parallel and perpendicular to the line of reference,

$$nmk^2 \cos\theta \cdot \omega = mk^2(\omega_1 \cos\beta_1 + \omega_2 \cos\beta_2 + \cdots \omega_n \cos\beta_n) = mk^2A;$$

$$nmk^2 \sin\theta \cdot \omega = mk^2(\omega_1 \sin\beta_1 + \omega_2 \sin\beta_2 + \cdots \omega_n \sin\beta_n) = mk^2B,$$

where A and B are the values respectively of the parenthetical quantities. Reducing, we have

$$\omega = \frac{\sqrt{A^2 + B^2}}{n},$$

$$\theta = \cos^{-1}\frac{A}{\sqrt{A^2+B^2}} = \sin^{-1}\frac{B}{\sqrt{A^2+B^2}}.$$

22. *A prismatic bar rotating in free space suddenly snaps asunder at its centre; required the subsequent motion.*

The body will rotate about its centre, and after separation each half will rotate about its own centre, and those centres will have the same uniform velocity that they had immediately preceding separation. After separation there will be two independent systems, still at the instant of separation the entire moment of the momentum will equal that of the original bar.

Let l be the length of bar, m its mass, k_1 its principal radius of gyration, ω_1 its angular velocity, ω the required angular velocity of each half after separation, v the velocity of each half, and k the principal radius of gyration of each half; then we have

$$mk_1^2\omega_1 = \tfrac{1}{2}mk^2\omega + \tfrac{1}{2}mk^2\omega + mv \cdot \tfrac{1}{4}l$$

$$= mk^2\omega + \tfrac{1}{4}mlv ;$$

also

$$k_1^2 = \tfrac{1}{12}l^2, \quad k^2 = \tfrac{1}{12}(\tfrac{1}{2}l)^2 = \tfrac{1}{48}l^2, \quad v = \tfrac{1}{4}l\omega ;$$

which, substituted in the preceding, gives

$$\omega = \omega_1 ;$$

hence *The angular velocity of each half after separation will be the same as that of the original bar, and the two halves will move in opposite directions with a uniform velocity.* Since no forces are conceived to act upon the bar, the kinetic energy after separation will be the same as before ; hence we would have, equation (154),

$$2(\tfrac{1}{4}mv^2) + \tfrac{1}{2}mk^2\omega^2 = \tfrac{1}{2}mk_1^2\omega_1^2 ;$$

or

$$(\tfrac{1}{4}l\omega)^2 + \tfrac{1}{12}(\tfrac{1}{2}l)^2\omega^2 = \tfrac{1}{12}l_1^2\omega_1^2 ;$$

$$\therefore \omega = \omega_1,$$

as before.

Suppose that such a bar separates into n equal parts, what will be the subsequent motion ? Or if it suddenly melts, or dissolves, will the elements partake of the rotary motion ? If two or more bodies having a rotary motion cohere, will the aggregate mass have a rotary motion ? Can rotary motion impart a motion of translation ?

23. If a bar rotating about one end suddenly snaps asunder, required the subsequent motion.

24. If a bar rotating about one end gradually melts away at the free end, will the angular velocity of the remaining part be changed ?

25. *A spherical shell of infinitesimal thickness, mass m, radius r, is filled with a frictionless fluid, mass m' ; the shell is rotating with an angular velocity ω when the fluid becomes solid and rotates with the shell ; required the common angular velocity of the mass.*

Let $\omega' =$ the required angular velocity. The moment of the momentum of the shell when the included mass is a fluid, will

be $\frac{2}{5}mr^2\omega$, and of the entire mass after it becomes one solid will be $\frac{2}{5}mr^2\omega' + m'k^2\omega'$;

$$\therefore\ \tfrac{2}{5}mr^2\omega = (\tfrac{2}{5}mr^2 + m'k^2)\omega';$$

$$\therefore\ \omega' = \frac{\frac{2}{5}mr^2}{\frac{2}{5}mr^2 + m'k^2}\,\omega.$$

If $k^2 = \frac{2}{5}r^2$, then

$$\omega' = \frac{5m}{5m + 3m'}\,\omega.$$

If $m' = 0$,

$$\omega' = \omega,$$

as it should.

26. *A spherical shell, external radius r, internal radius r_1, mass m, filled with a frictionless fluid, mass m', rolls on a perfectly rough horizontal plane with a velocity v; the fluid freezes and rolls with the shell; required the velocity v' of the common mass.*

Let ω be the angular velocity before freezing, and ω' after. The moment of the momentum before freezing will be

$$mk^2\omega + (m + m')v \cdot r,$$

and this will equal the moment of the momentum after freezing, hence

$$mk^2\omega + (m + m')vr = mk^2\omega' + m'k_1^2\omega' + (m + m')v'r,$$

also

$$v = r\omega, \quad v' = r\omega', \quad k_1^2 = \tfrac{2}{5}r_1^2, \quad k^2 = \frac{2}{5}\,\frac{r^5 - r_1^5}{r^3 - r_1^3};$$

$$\therefore v' = \frac{2m(r^5 - r_1^5) + 5(m + m')(r^3 - r_1^3)r^2}{2m\,(r^5 - r_1^5) + 2m'r_1^2\,(r^3 - r_1^3) + 5\,(m + m')\,(r^3 - r_1^3)r^2}\,v.$$

If $m' = 0$, we have $v' = v$, as we should. The kinetic energy of the entire mass after freezing will be less than before; and, generally, whenever the internal forces cause a *relative* change of parts or particles of the system, the kinetic energy *may* be changed.

27. *A circle is revolving freely about a diameter with the angular velocity ω, when a point in the extremity of the perpen-*

dicular diameter becomes suddenly fixed; required the instantaneous angular velocity ω'.

We have

$$mk_1^2\,\omega = mk^2\omega';$$

$$\therefore\ \omega' = \frac{k_1^2}{k^2}\,\omega = \frac{\frac{1}{4}r^2}{\frac{5}{4}r^2}\,\omega = \tfrac{1}{5}\omega.$$

28. *A cone revolves about its axis with an angular velocity ω; the altitude contracts, the volume remaining constant, required the resultant angular velocity.*

Let h be the original altitude, x any subsequent altitude, r the original radius, y the radius when the altitude is x; then, the volume being constant,

$$\tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi y^2 x,$$

the moment of inertia will be

$$mk^2 = \tfrac{1}{10}\pi r^4 h,$$

$$mk_1^2 = \tfrac{1}{10}\pi y^4 x;$$

but the moment of the momentum being constant,

$$mk_1^2\,\omega = mk^2\omega_1;$$

$$\therefore\ \omega_1 = \frac{k_1^2}{k^2}\,\omega = \frac{r^2}{y^2}\,\omega = \frac{x}{h}\,\omega,$$

or the angular velocity will vary directly as the altitude of the cone.

29. *One end of a fine, inextensible string is attached to a fixed point, and the other end to a point in the surface of a homogeneous sphere, and the ends brought together, the centre of the sphere being in a horizontal through the ends of the string, and the slack string hanging vertically. The sphere is let fall and an angular velocity imparted to it at the same instant, the sphere winding up the string on the circumference of a great circle until it winds up all the slack, when it suddenly begins to ascend, winding up the string, the sphere returning just to the*

starting point. Required the initial angular velocity, the tension of the string during the ascent of the sphere, the initial upward velocity of the centre of the sphere, and the time of movement.

Let $l =$ the length of the string, $r =$ radius of the sphere, $m =$ its mass, $k =$ its radius of gyration, $x =$ the length of the unwound portion of string at the end of descent, $v =$ velocity at the end of descent, v' the velocity with which the sphere begins to ascend, $\omega =$ the initial angular velocity, $\omega' =$ the angular velocity with which the sphere begins to ascend, $t_1 =$ the time of descent, t_2 the time of ascent, $T =$ tension of the string during the ascent, and I the impulse communicated by the string to the sphere at the end of descent.

The length of string wound up at end of descent is

$$r\omega t_1 = r\omega\sqrt{\frac{2x}{g}},$$

t_1 being the time of falling freely through the height x,

$$\therefore\ x = l - r\omega\sqrt{\frac{2x}{g}},$$

whence

$$\omega = \frac{l - x}{r}\sqrt{\frac{g}{2x}}. \tag{1}$$

For the impulsive motion,

$$\omega - \omega' = \frac{Ir}{mk^2},$$

$$v + v' = \frac{I}{m}.$$

Eliminating I,

$$\frac{k^2}{r}(\omega - \omega') = v + v'.$$

But $k^2 = \frac{2}{5}r^2$, and $v = \sqrt{2gx}$,

$$\therefore\ \omega' = \omega - \frac{5}{2r}(\sqrt{2gx} + v'). \tag{2}$$

The upward motion, the origin being at the point where the centre begins to ascend, and the axis of y positive upwards,

$$m\frac{d^2y}{dt^2} = T - mg. \tag{3}$$

For the angular acceleration,

$$mk^2\frac{d^2\theta}{dt^2} = -rT,$$

or

$$m\frac{d^2\theta}{dt^2} = -\frac{5T}{2r}. \tag{4}$$

Also

$$y = r\theta,$$

or

$$\frac{d^2y}{dt^2} = \frac{rd^2\theta}{dt^2}. \tag{5}$$

Eliminating $\frac{d^2y}{dt^2}$ and $\frac{d^2\theta}{dt^2}$ from (3), (4), (5), $T = \frac{2}{7}mg = \frac{2}{7}$ the weight of the sphere. Eliminating T and $\frac{d^2\theta}{dt^2}$ from (3), (4), (5), $\frac{d^2y}{dt^2} = -\frac{5}{7}g$. Integrating, observing that when $t = 0$, $\frac{dy}{dt} = v'$,

$$\frac{dy}{dt} = v' - \frac{5}{7}gt. \tag{6}$$

When $\frac{dy}{dt} = 0$, $t = t_2$;

$$\therefore v' = \frac{5}{7}gt_2. \tag{7}$$

Integrating (6), observing that when $t = 0$, $y = 0$,

$$y = v' \cdot t - \frac{5}{14}gt^2.$$

When $y = x$, $t = t_2$,

$$x = v't_2 - \frac{5}{14}gt_2^2. \tag{8}$$

From (7), (8),

$$v' = \sqrt{\tfrac{10}{7}gx}, \tag{9}$$

$$t_2 = \sqrt{\frac{14x}{5g}}. \tag{10}$$

From (5),

$$\frac{dy}{dt} = r\frac{d\theta}{dt},$$

$$\therefore v' = r\omega', \quad \omega' = \frac{1}{r}\sqrt{\tfrac{10}{7}gx}. \tag{11}$$

Substituting these values of v' and ω' in (2),

$$x = l(6 - \sqrt{35}).$$

Substituting this value of x in (1) and (9),

$$\omega = \frac{(\sqrt{35} - 5)\sqrt{gl}}{r\sqrt{[2(6 - \sqrt{35})]}},$$

$$v' = \sqrt{[\tfrac{10}{7}gl(6 - \sqrt{35})]},$$

$$t_1 = \sqrt{\frac{2x}{g}} = \sqrt{\left(\frac{2l(6 - \sqrt{35})}{g}\right)}.$$

From (10),

$$t_2 = \sqrt{\frac{14x}{5g}} = \sqrt{\left(\frac{14l(6 - \sqrt{35})}{5g}\right)}.$$

The whole time is

$$t_1 + t_2 = (1 + \tfrac{1}{5}\sqrt{35})\sqrt{\left(\frac{2l(6 - \sqrt{35})}{g}\right)}.$$

(Problem by the Author in *Mathematical Visitor*, Jan., 1879.)

30. *If n concentric uniform spherical shells of infinitesimal thickness moving without friction about axes whose inclinations to three rectangular axes are α_1, β_1, γ_1; α_2, β_2, γ_2, etc., with angular velocities ω_1, ω_2, etc., respectively, suddenly become one*

solid; required the resultant angular velocity and resultant axis.

Let α, β, γ, be the direction-angles, ω the resultant angular velocity, k the radius of gyration, and m the mass of each; then, Art. 166,

$$nmk^2\omega \cos \alpha = mk^2(\omega_1 \cos \alpha_1 + \omega_2 \cos \alpha_2 +, \text{etc.}) = A, \text{ (say)}.$$

$$nmk^2\omega \cos \beta = mk^2(\omega_1 \cos \beta_1 + \omega_2 \cos \beta_2 +, \text{etc.}) = B,$$

$$nmk^2\omega \cos \gamma = mk^2(\omega_1 \cos \gamma_1 + \omega_2 \cos \gamma_2 +, \text{etc.}) = C;$$

and from Coördinate Geometry, Art. 198, Eq. (3),

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Substituting from the preceding, we have

$$A^2 + B^2 + C^2 = n^2m^2k^4\omega^2;$$

$$\therefore \omega = \frac{\sqrt{A^2 + B^2 + C^2}}{nmk^2}.$$

This value in each of the preceding values gives

$$\cos \alpha = \frac{A}{\sqrt{A^2 + B^2 + C^2}},$$

$$\cos \beta = \frac{B}{\sqrt{A^2 + B^2 + C^2}},$$

$$\cos \gamma = \frac{C}{\sqrt{A^2 + B^2 + C^2}}.$$

If the motion be in the plane xy, we will have $\gamma_1 = 90° = \gamma_2 = \gamma_3$, etc.; hence $\gamma = 90°$, and the third of the preceding equations vanishes, and the case reduces to the problem given by the author in the *Mathematical Visitor*, Jan., 1882, p. 14.

31. A screw of Archimedes is free to turn about its axis placed vertically; a particle placed at the upper end of the tube runs down through it; determine the resultant angular velocity imparted to the tube.

Let m = the mass of the particle,
nm = the mass of the screw,
a = radius of the screw,
z = the vertical distance the particle has descended,
and, θ and φ = the angles through which the screw and particle have respectively revolved about the axis in the time t.

The energy of both bodies equals that imparted by gravity, or

$$ma^2 \frac{d\varphi^2}{dt^2} + m \frac{dz^2}{dt^2} + mna^2 \frac{d\theta^2}{dt^2} = 2mgz. \tag{1}$$

From the principle of Conservation of Areas, we have

$$ma^2 \frac{d\varphi}{dt} = mn\, a^2 \frac{d\theta}{dt}. \tag{2}$$

We also have the geometrical equation

$$z = a\,(\varphi + \theta) \tan \alpha. \tag{3}$$

From (3),

$$\frac{dz^2}{dt^2} = a^2 \tan^2 \alpha \left(\frac{d\varphi^2}{dt^2} + 2\frac{d\varphi}{dt} \cdot \frac{d\theta}{dt} + \frac{d\theta^2}{dt^2}\right). \tag{4}$$

Substituting in (1),

$$a^2\left(\frac{1}{\cos^2 \alpha} \cdot \frac{d\varphi^2}{dt^2} + 2\frac{\sin^2 \alpha}{\cos^2 \alpha} \cdot \frac{d\varphi}{dt} \cdot \frac{d\theta}{dt} + \frac{\sin^2 \alpha}{\cos^2 \alpha}\frac{d\theta^2}{dt^2} + n\frac{d\theta^2}{dt^2}\right) = 2gz. \tag{5}$$

From (2),

$$\frac{d\varphi}{dt} = n\frac{d\theta}{dt}.$$

Substituting in (5),

$$a^2\left(\frac{n^2}{\cos^2 \alpha} + \frac{2n \sin^2 \alpha}{\cos^2 \alpha} + \frac{\sin^2 \alpha}{\cos^2 \alpha} + n\right)\frac{d\theta^2}{dt^2} = 2gz,$$

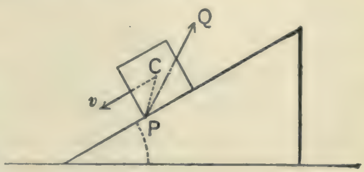
or,

$$a^2(n+1)(n + \sin^2 \alpha)\frac{d\theta^2}{dt^2} = 2gz \cos^2 \alpha.$$

When $\dfrac{d\theta}{dt} = \omega,\ z = h,$

$$\therefore\ \omega = \sqrt{\frac{2gh\cos^2\alpha}{(n+1)(n+\sin^2\alpha)}}.$$

32. *A cube slides down an inclined plane with four of its edges horizontal. The middle point of its lowest edge comes in contact with a small fixed obstacle; determine the limiting velocity that the cube may be on the point of overturning.*



Let v be the velocity of the cube at the instant of impact, $2b$ the length of one edge of the cube, k the radius of gyration in reference to the edge at P, and k_1 the principal radius of gyration in reference to a parallel axis, ω the initial angular velocity, and m the mass of the cube.

The moment of the momentum just before impact will be

$$mv \cdot b. \tag{1}$$

The initial moment of the momentum after impact, will be, equations (155) and (123), and example 4, page 172,

$$Qa = mk^2\omega = m(2b^2 + k_1^2)\omega = m(2b^2 + \tfrac{2}{3}b^2)\omega$$

$$= \tfrac{8}{3}mb^2\omega. \tag{2}$$

Hence, Article 166,

$$mvb = \tfrac{8}{3}mb^2\omega;$$

$$\therefore\ v = \tfrac{8}{3}b\omega. \tag{3}$$

The cube will be on the point of overturning when the energy due to rotary velocity is just sufficient to raise the centre to a point vertically over P. The energy will be, equations (153) and (123),

$$\tfrac{1}{2}\Sigma mr \cdot \omega^2 = \tfrac{1}{2}mk^2 \cdot \omega^2 = \tfrac{1}{2}m\omega^2\tfrac{8}{3}b^2 = \tfrac{4}{3}mb^2\omega^2. \tag{4}$$

The work of raising the centre to its highest point will be

$$mg \cdot b\sqrt{2}[1 - \cos(\tfrac{1}{4}\pi - \beta)], \tag{5}$$

which being equal to (4), we have

$$\omega^2 = \tfrac{3}{4}\sqrt{2}\,\frac{g}{b}\,[1 - \cos(\tfrac{1}{4}\pi - \beta)], \tag{6}$$

and this substituted in (3) gives

$$v^2 = \tfrac{16}{3}\sqrt{2} \cdot bg[1 - \cos(\tfrac{1}{4}\pi - \beta)], \tag{7}$$

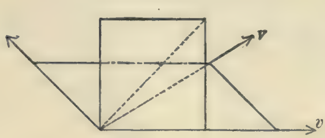
which is the required result.

In this problem we may consider two impulses, one that of the momentum before impact; the other that destroyed by the impact. It is now required to find the magnitude and direction of a single impulse, which applied at P, will produce the same effect. At first it seems that this impulse will be parallel to the plane and opposed to the direction of motion, but if the body were free, it would, in this case, rotate about its centre, equations (168), in which case the corner at P would move perpendicularly to the diagonal through the centre, whereas, in the problem, this corner becomes instantaneously fixed. The resultant impulse may be considered as the resultant of an impulse acting along the plane, and another acting at P perpendicular to the diagonal of the centre through that point. The angular velocity is, equation (3),

$$\omega = \frac{3}{8}\,\frac{v}{b}; \tag{8}$$

hence the actual velocity, u, of the centre is

$$\sqrt{2}b \cdot \omega = u = \tfrac{3}{8}\sqrt{2}v;$$



which is also the initial velocity with which the point P would move normally to the diagonal if the body were free. The plane and obstacle then, impose the two component velocities v and u, the angle between which is $\tfrac{3}{4}\pi$; hence the resultant velocity V will be

$$V^2 = v^2 + u^2 + 2vu \cos \tfrac{3}{4}\pi,$$

$$= v^2 + \tfrac{9}{32}v^2 - \tfrac{6}{8}v^2;$$

$$\therefore \; V = \tfrac{1}{8}\sqrt{34}v. \tag{9}$$

To find the angle between V and v, we have

$$u^2 = V^2 + v^2 - 2vV \cos \varphi;$$

$$\therefore \; \cos \varphi = \frac{5}{\sqrt{34}}, \tag{10}$$

which is constant, as we might have anticipated from (9), since V varies directly as v. To find the energy lost by the impact, we have, for the kinetic energy before impact, equation (24),

$$\tfrac{1}{2}mv^2; \tag{11}$$

and for the initial energy after impact, equation (153), problem 4, page 172, equation (123), and (8) above

$$\tfrac{1}{2}\Sigma mr^2\omega^2 = \tfrac{1}{2}mk^2\omega^2 = \tfrac{1}{2}m(2b^2 + k_1^2)\left(\frac{3}{8}\frac{v}{b}\right)^2$$

$$= \tfrac{3}{16}mv^2, \tag{12}$$

which is independent of the dimensions of the cube, the mass remaining constant. This is only $\frac{3}{8}$ of the energy in the cube before impact, hence $\frac{5}{8}$ will have been removed and passed into heat.

33. *A circular disc, radius r, mass m, rolling on a rough horizontal plane with a velocity v, impinges against an obstacle whose height is b; if there be no slipping on the obstacle, required the velocity immediately after impact, the height to which the centre of the disc will be raised if it does not pass over the obstacle, and the velocity at the highest point if it does pass over, and the velocity of approach that it may just roll over the obstacle.*

The velocity immediately after impact may be found as in Article 145, and the other results as in the preceding problem.

34. *A uniform bar of infinitesimal section, length l, mass m, moving with a uniform velocity v, strikes an equal bar at rest but free to move ; if the two bars are mutually perpendicular, and the former moves in a path perpendicular to both, required the motions of each after impact, supposing the end of one is struck by the end of the other, both bodies being inelastic. What will be their subsequent motions if they rigidly adhere at the instant of impact?*

Let OA be the position of the moving rod, OB that of the rod at rest, at the instant of impact; OX the direction in which O was moving, u the velocity with which the centre, C, of OA was moving before impact, v its velocity after impact in the direction OX, v' the velocity with which the centre D of the other rod begins to move; k the radius of gyration of each rod about its centre, ω, ω', their respective angular velocities about their centres; Q the impulsive reaction at O. Put $\frac{1}{2}l = a$, we have

$$mv = mu - Q. \tag{1}$$

$$mk^2\omega = Qa. \tag{2}$$

$$mv' = Q. \tag{3}$$

$$mk^2\omega' = Qa. \tag{4}$$

The velocity at O of the moving rod after impact is $v - a\omega$; that of the other rod is $v' + a\omega'$. Since the rods are inelastic, these velocities are equal, hence

$$v - a\omega = v' + a\omega'. \tag{5}$$

These five equations readily give

$$v = \tfrac{1}{2}u\left(\frac{2a^2 + k^2}{a^2 + k^2}\right), \quad v' = \tfrac{1}{2}u\left(\frac{k^2}{a^2 + k^2}\right).$$

$$\omega = \omega' = \tfrac{1}{2}u\left(\frac{a}{a^2 + k^2}\right).$$

If the rods rigidly adhere at the time of contact, the values of v, v', ω, ω', found above, will act after contact on the connected rods. Join CD; its middle point, G, will be the centre of gravity of the connected rods. Join OG. Let $V=$ the velocity of G after impact in the direction OX, $2mV = mu$; $\therefore\ V = \frac{1}{2}u$. The velocity of O in the same direction is $v - a\omega = \frac{1}{2}u$. Hence OG will move in the direction OX with a velocity $= \frac{1}{2}u$.

$$v - V = \tfrac{1}{2}u \cdot \frac{a^2}{a^2 + k^2},$$

$$V - v' = \tfrac{1}{2}u \cdot \frac{a^2}{a^2 + k^2}.$$

Hence C and D will both begin to revolve about G with a velocity $= \frac{1}{2}u \dfrac{a^2}{a^2+k^2}$. The angular velocity of the system about $OG = \dfrac{u}{CD} \times \dfrac{a^2}{a^2+k^2} = \dfrac{u}{\sqrt{2}} \cdot \dfrac{a}{a^2+k^2}$. Substituting $k^2 = \frac{1}{3}a^2$; $a = \frac{1}{2}l$; $v' = \frac{1}{8}u$, $\omega = \omega' = \dfrac{3}{4}\dfrac{u}{l}$, and the angular velocity about OG, when the rods adhere, will be

$$\frac{3}{2}\,\frac{u}{l\sqrt{2}}.$$

FRICTION.

35. *Find the conditions that a hoop shall roll down an inclined plane without sliding, μ being the coefficient of friction, and i the inclination of the plane.*

Let θ be the angle through which the hoop has rolled at the time t, a the radius, m the mass, and F the friction. Then we have, Art. 125,

$$a^2 m \frac{d^2\theta}{dt^2} = Fa,$$

and

$$m\frac{d^2(a\theta)}{dt^2} > gm \sin i - F.$$

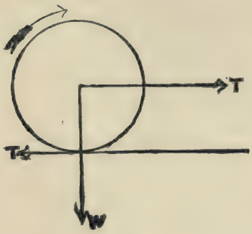
therefore

$$gm \sin i < 2F;$$

but $$F = gm\mu \cos i;$$

therefore $$\tan i < 2\mu.$$

36. *A sphere having a rotary motion about a horizontal axis is placed gently on a rough plane; determine its motion.*



Let $W =$ the weight of the sphere, μ the coefficient of friction, T the tangential action due to friction.

There may be two cases: 1st, if there be slipping; and 2d, if there be no slipping. In the former case $T = \mu W$.

For the motion of the centre we have

$$\left.\begin{aligned} M\frac{d^2x}{dt^2} &= T = \mu W, \\ \text{or} \qquad \frac{d^2x}{dt^2} &= \mu g \end{aligned}\right\}; \qquad (1)$$

and for the rotary motion,

$$\left.\begin{aligned} Mk^2\frac{d^2\varphi}{dt^2} &= -Tr = -\mu Wr, \\ \text{or} \qquad k^2\frac{d^2\varphi}{dt^2} &= -\mu gr \end{aligned}\right\}. \qquad (2)$$

Integrating, we have

$$\left.\begin{aligned} \frac{dx}{dt} &= \mu gt + c_1 \\ x &= \tfrac{1}{2}\mu gt^2 + c_1t + c_2 \\ k^2\frac{d\varphi}{dt} &= -\mu grt + c' \\ k^2\varphi &= -\tfrac{1}{2}\mu grt^2 + c't + c'' \end{aligned}\right\}. \qquad (3)$$

For $t = 0$, we have $x = 0$, and $\varphi = 0$ $\therefore$ $c_2 = 0$, and $c'' = 0$; also for $t = 0$, $\dfrac{d\varphi}{dt} = \omega_0$, the initial angular velocity, $\therefore$ $c' = k^2\omega_0$;

and $\frac{dx}{dt} = 0$, the initial velocity of the centre, and the corrected equations become

$$\left.\begin{aligned} \frac{dx}{dt} &= \mu g t \\ x &= \tfrac{1}{2}\mu g t^2 \\ \frac{d\varphi}{dt} &= \omega_0 - \frac{\mu g r t}{k^2} \\ \varphi &= \omega_0 t - \frac{1}{2}\frac{\mu g r t^2}{k^2} \end{aligned}\right\}. \qquad (4)$$

The velocity of the point of contact will be plus $\frac{dx}{dt}$, and minus $r\frac{d\varphi}{dt}$; hence

$$\frac{dx}{dt} - r\frac{d\varphi}{dt} = \mu g t - \omega_0 r + \frac{\mu g r^2 t}{k^2}; \qquad (5)$$

and so long as this is finite the preceding equations will hold true, but when it reduces to zero the conditions change. To find when they change, make each member of (5) equal to zero, and solve for t, which call t_1; hence

$$t_1 = \frac{\omega_0 r k^2}{\mu g(k^2 + r^2)}. \qquad (6)$$

The left member of (5) becomes

$$\frac{dx}{dt} = r\frac{d\varphi}{dt}; \qquad (7)$$

integrating,

$$x = r\varphi; \qquad (8)$$

differentiating,

$$\frac{d^2x}{dt^2} = r\frac{d^2\varphi}{dt^2}; \qquad (9)$$

which are the equations of motion when there is no slipping. The value of T is not equal to μW after the time t_1, and to find its value combine the first of (1) and (2) with (9), giving

$$\frac{d^2x}{dt^2} - r\frac{d^2\varphi}{dt^2} = T\left(\frac{k^2 + r^2}{k^2}\right) = 0\,;$$

$$\therefore\ T = 0\,; \tag{10}$$

hence it requires no friction to cause the point of contact to have no progressive motion. The motion is the same as if the body were in void space under the action of no forces, having a uniform motion both as to the translation of the centre, and of rotation about the centre.

The total amount of slipping will be

$$r\varphi - x = r\omega_0 t_1 - \tfrac{1}{2}\mu g\,\frac{k^2 + r^2}{r^2}\,t_1^2 \tag{11}$$

37. If a sphere, radius 3 feet, weight 20 pounds, rotating ten times per second, be placed on a horizontal plane whose coefficient of friction is $\frac{1}{10}$; how long will it be in coming to a uniform velocity, how far will it have traveled, how much will it have slipped, what will be the uniform velocity of the centre, and the uniform angular velocity?

38. If a cylinder have the same amount of material, diameter, and rate of rotation, as the sphere in the preceding example, and placed on the same plane, which will first attain a uniform motion, which will have the greater uniform velocity, and which will have slipped most?

39. If a cylinder whose altitude equals the diameter of the sphere of example 37, the same amount of material and rate of rotation, be placed on the same plane, which will finally attain the greater uniform rotation?

40. What must be the coefficient of friction that there be no slipping at the point of contact at the beginning of motion?

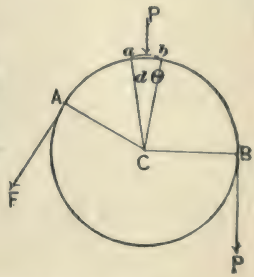
41. Which will first attain a uniform motion, the sphere in example 37, or a sphere of the same material and twice the diameter?

42. If the body gradually contracts, retaining a constant mass and same form, will it go farther or not before attaining a uniform velocity?

43. If a sphere preserves the same circle of contact, but gradually contracts laterally, changing to an oblate ellipsoid, will it

affect the time of attaining a uniform velocity? What will be the time if the polar axis becomes half the equatorial diameter?

44. *A rope is stretched round a rough cylindrical surface subtending an angle θ, the coefficient of friction being μ; required the force F acting in the direction of the tangent at one end in order that P will be in a state just bordering on motion towards F.*



Let the tension at any point, a, be t, that adjacent, b, will be $t + dt$, p the normal pressure on the arc per unit of length if it were uniform; hence, on an element of length, it will be pds, μ the coefficient of friction; then

$$dt = \mu \cdot pds,$$

also

$$pds = \sqrt{t^2 + (t + dt)^2 + 2t(t + dt)\cos(\pi - \theta)}$$

$$= t\sqrt{2(1 + \cos(\pi - \theta))}, \text{ (ultimately)}$$

$$= t \cdot 2\cos \tfrac{1}{2}(\pi - \theta) = t \cdot 2 \sin \tfrac{1}{2}\theta$$

$$= td\theta \text{ (ultimately)};$$

$$\therefore\ dt = \mu t \cdot d\theta.$$

Integrating,

$$\log t = \mu\theta + C.$$

But for

$$\theta = 0, \quad t = F,$$

and for

$$\theta = AB, \quad t = P;$$

$$\therefore\ P = F\varepsilon^{\mu\theta}.$$

From the relations

$$pds = td\theta,$$

and

$$ds = rd\theta,$$

we find

$$p = \frac{t}{r},$$

the same as equation (o), p. 139.

45. A shaft having a bearing the entire length is driven by a pulley at one end, the power being taken at the other. Find the diameter of the shaft at any point for uniform strength.

Let $w =$ the weight of the shaft per unit volume, $\mu =$ the coefficient of friction, $h =$ the radius at the driving end, x the distance of any cross section from the driving end, and y the radius of that cross section. Then

$$w\mu\pi y^2 dx = \text{friction of an element,}$$

$$w\mu\pi y^3 dx = \text{moment of friction.}$$

Then

$$\int w\mu\pi y^3 dx + PR = cy^3, \qquad (a)$$

PR being the moment of the driving power. From (a)

$$w\mu\pi y^3 dx = 3cy^2 dy,$$

or, letting

$$\frac{w\mu\pi}{3c} = A,$$

$$\int_0^x A dx = \int_y^h \frac{dy}{y},$$

$$Ax = \log\frac{h}{y},$$

and

$$y = h\varepsilon^{-Ax}.$$

ATTRACTION.

46. *Assume that two spheres of the same material as the earth, each one foot in diameter, are reduced in size to a mere point at their centres, and placed one foot from each other, required the time it would take for them to come together by their mutual attraction, they being uninfluenced by any external force.*

Let $E =$ the mass of the earth,
$m =$ the mass of one of the spheres,
$m' =$ the mass of the other sphere,
$R =$ the radius of the earth,
$r =$ the radius of one of the spheres,

r' = the radius of the other sphere,
g = the acceleration due to gravity on the earth,
μ = the acceleration due to the attraction of a sphere of mass unity, upon another sphere of mass unity, the distance between their centres being unity,
a = the original distance between the centres of m and m', and
x = the distance between their centres at the end of time t.

Then from (227) we have

$$\frac{d^2x}{dt^2} = -(m + m')\frac{R^2g}{E}\cdot\frac{1}{x^2},$$

which integrated (pp. 33, 34), observing that for $t = 0$, $x = a$, and $v = 0$, and that μ in the reference equals $(m + m')\dfrac{R^2g}{E}$ in this case, gives

$$t = \left[\frac{Ea}{2(m + m')R^2g}\right]^{\frac{1}{2}} \times \left[(ax - x^2)^{\frac{1}{2}} + a\cos^{-1}\left(\frac{x}{a}\right)^{\frac{1}{2}}\right]_a^0,$$

which for the limits gives

$$t = \tfrac{1}{2}\pi a\left[\frac{Ea}{2(m + m')R^2g}\right]^{\frac{1}{2}}.$$

If both spheres are of the same density, their masses will be as the cubes of their radii ; or

$$m = \frac{r^3}{R^3}E,$$

$$m' = \frac{r'^3}{R^3}E;$$

and we have

$$t = \tfrac{1}{2}\pi a\left[\frac{Ra}{2(r^3 + r'^3)g}\right]^{\frac{1}{2}};$$

and if the spheres are equal, as in the problem, we have

$$t = \tfrac{1}{4}\pi a\left(\frac{Ra}{r^3g}\right)^{\frac{1}{2}}.$$

If $a = 1$ foot, $R = 20{,}850{,}000$, $r = \frac{1}{2}$ foot, $g = 32\frac{1}{6}$, we have

$$t = .7854\left(\frac{166{,}800{,}000}{32.166}\right)^{\frac{1}{2}},$$

$$= 1{,}788 \text{ seconds, nearly,}$$

$$= 29.8 \text{ minutes, nearly.}$$

47. *To find the stress in pounds which would be exerted by the mutual action of two such spheres as in example* 46, *at a distance of one foot between their centres, we have, from equations* (224) *and* (235), *since* $m = m'$, *and* $x = 1$,

$$\frac{1}{64}\frac{Eg}{R^4}.$$

The mass of the earth is $5\frac{1}{2}$ times an equal mass of water. The weight of a cubic foot of water is $62\frac{1}{2}$ lbs. at the place where $g = 32\frac{1}{6}$, and the volume of the earth is $\frac{4}{3}\pi R^3$, hence

$$E = 5\tfrac{1}{2} \times 62\tfrac{1}{2} \times \tfrac{4}{3}\pi R^3,$$

which, substituted above, gives for the stress

$$\frac{\frac{4}{3}\pi \times 5\frac{1}{2} \times 62\frac{1}{2} \times 32\frac{1}{6}}{64 \times 20{,}850{,}000}$$

$$= .00003471 + \text{lb.}$$

or nearly $\dfrac{347}{10{,}000{,}000}$ of a pound, a quantity inappreciably small.

48. *If the density of the earth at the surface be unity, and at the centre is* m, g *the force of gravity at the surface, and* f *that at any distance* x *from the centre within the earth, what is the law for* f, *the density increasing uniformly toward the centre.*

The density at any distance, x, from the centre will be

$$\frac{(m-1)}{r}(r - x) + 1.$$

The volume of a sphere of radius x is $4\pi \int_0^r x^2 dx$. Hence we have,

$$Mass = 4\pi \int_0^x \left[\frac{(m-1)}{r}(r-x) + 1 \right] x^2 dx,$$

and

$$f = c\frac{4\pi}{x^2}\left[\frac{(m-1)}{r}\left(\frac{rx^3}{3} - \frac{x^4}{4}\right) + \frac{x^3}{3}\right]$$

$$= c\frac{\pi}{3r}\left[4(m+r-1)x - 3(m-1)x^2\right].$$

If

$$x = r, \quad f = g,$$

hence,

$$g = c\frac{\pi}{3r}\left[4(m+r-1)r - 3(m-1)r^2\right];$$

$$\therefore f = \frac{4(m+r-1)x - 3(m-1)x^2}{4(m+r-1)r - 3(m-1)r^2} \cdot g.$$

49. *Considering the earth and moon as uniform bodies, the mass of the earth* $5\frac{1}{2}$ *times that of an equal volume of water, and its radius* 3,956 *miles, the mass of the moon* $3\frac{1}{2}$ *times that of an equal volume of water, and its radius* 1,080 *miles, the mean distance between the centres of the earth and moon* 60 *times the radius of the earth, and the acceleration due to gravity at the surface of the earth* $32\frac{1}{6}$ *feet per second; required the time in which they would come together by their mutual attraction.*

When they are in contact the distance between their centres will be 5,036 miles, and we have from Problem 46,

$$t = \left[\frac{Ma}{2(M+m)R^2 g}\right]^{\frac{1}{2}} \left[(ax - x^2)^{\frac{1}{2}} + a\cos^{-1}\left(\frac{x}{a}\right)^{\frac{1}{2}}\right]_{237,360}^{5,036}. \quad (1)$$

But

$$\frac{7M}{11m} = \frac{R^3}{r^3}, \quad \text{or,} \quad m = \frac{7}{11}\frac{Mr^3}{R^3},$$

and

$$t = \left[\frac{Ra}{2(R^3 + \frac{7}{11}r^3)g}\right]^{\frac{1}{2}} \left[(ax - x^2)^{\frac{1}{2}} + a\cos^{-1}\left(\frac{x}{a}\right)^{\frac{1}{2}}\right]_{237,360}^{5,036}. \quad (2)$$

Substituting numerical values in equation (2),

$$t = \left[\frac{3{,}956 \times 237{,}360 \times 5{,}280 \times 3}{193[(3{,}956)^3 + \frac{7}{11}(1{,}080)^3]}\right]^{\frac{1}{2}} \left[\left(237{,}360 \times 5{,}036 - (5{,}036)^2\right)^{\frac{1}{2}}\right.$$

$$\left. + \, 237{,}360 \cos^{-1}\left(\frac{5{,}036}{237{,}360}\right)^{\frac{1}{2}}\right]$$

$$= 412{,}945 \text{ seconds.}$$

Loomis gives 415,600 seconds as the time required for a particle to fall from the moon to the earth, the distance to the moon being the same as that given in this example, which is 2,655 seconds, or nearly 44 minutes more than the time for the earth and the moon to meet.

50. *When the earth is in perihelion, suppose the sun's mass to be increased by x times its present value. Required the change in the elements of the terrestrial orbit.*

The square of the eccentricity of any planetary orbit is (Problem 4, page 189),

$$1 - 2\,\frac{V^2 r \sin^2\beta}{\mu} + \frac{V^4 r^2 \sin^2\beta}{\mu^2},$$

in which V is the velocity of the planet at the point whose radius vector is r, β the angle between the curve at that point and r, and μ a measure of the attractive force. In this case $\beta = 90°$, and the equation becomes, by reduction,

$$\frac{V^2 r}{\mu} = 1 \pm e.$$

If now the mass of the sun is increased suddenly x times its present value, μ becomes $x\mu$, and the eccentricity of the new orbit will be e_1 (say), hence,

$$\frac{V^2 r}{x\mu} = 1 \pm e_1;$$

and

$$\pm e_1 = \frac{V^2 r}{x\mu} - 1,$$

$$= \frac{1}{x}(1 \pm e) - 1,$$

in which e_1 must be used when the right member of the equation is *positive*, and $-e_1$ when it is negative.

If $x < \frac{1}{2}(1 \pm e)$, $e_1 > 1$, or the orbit will be a hyperbola.
" $x = \frac{1}{2}(1 \pm e)$, $e_1 = 1$, " " " parabola.
" $x \left\{ \begin{matrix} > \frac{1}{2}(1 \pm e) \\ < (1 \pm e) \end{matrix} \right\}$, $e_1 < 1$, " " " an ellipse.
" $x = 1 \pm e$, $\pm e_1 = 0$ " " " a circle.
" $x > 1 \pm e$, $-e_1 > -1$, " " " an ellipse.

In all but the last the given and resulting perihelia coincide; but in the last the given perihelion will coincide with the new aphelion. $x = \frac{2}{3}$ and $e = 0.016784$, the eccentricity of the earth's orbit, we find $-e_1 = -0.32214$, or $-e_1 = 0.34452$; hence the present perihelion would become the aphelion point of the new orbit, and the new orbit would be an ellipse with the eccentricity 0.32214 or 0.34452. The mean distance would become

$$a = \frac{3r}{2(1 + r)} = 0.7436$$

of its present distance.

The time of rotation about the sun would be

$$T = \tfrac{3}{2}\sqrt{\left[\left(\frac{r}{1 + r}\right)^3\right]} \times 365\tfrac{1}{4} \text{ days} = 191.25 \text{ days.}$$

51. *Suppose infinitesimal aerolites equally distributed through all space, everywhere moving equally in all directions with a uniform and constant absolute velocity. The aggregate mass intercepted in a given time by a given stationary sphere is supposed to be known. Determine the effect upon the eccentricity of a spherical planet of given mass and volume moving in an eccentric orbit all of whose elements are known.* (*Math. Visitor*, July, 1880.)

If the velocity of the planet be less than that of the aerolites, the same mass will be intercepted as if the planet was at rest. Consider this case.

The change of the elements of the orbit will be due to two causes. 1st. The increase of the mass of the planet will increase

the attractive force between the sun and the planet. 2d. The aerolites will cause a direct resistance to the motion of the planet.

Let M be the mass of the sun, m' the initial mass of the planet, s the distance between them, k a constant; then will the acceleration of one body in reference to the other at the end of time t (the time in the problem being unity) be

$$\frac{k(M + m' + mt)}{s^2},$$

hence at distance unity,

$$\mu = k(M + m' + mt),$$

and

$$d\mu = kmdt.$$

For an elliptical orbit we have, page 190,

$$V_0^2 r_0^2 = \mu a(1 - e^2) = c,$$

a being the semi-transverse axis. Since the changes are small, consider two quantities only to vary contemporaneously. If e be constant, we find

$$da = -\frac{kmc}{\mu^2(1 - e^2)} dt;$$

therefore,

$$\Delta a = -\frac{kmct}{\mu^2(1 - e^2)} \text{ nearly.}$$

Similarly, if a be constant, we find

$$de = \frac{km(1 - e^2)}{2e\mu} dt;$$

therefore,

$$\Delta e = \frac{kmct}{2\mu \sqrt{a\mu\,(a\mu - c)}} \text{ nearly.}$$

Hence, the major axis decreases and the eccentricity increases with the time, and the amount of change for one revolution may be found by making t equal to the corresponding time.

2d. The law of resistance is not given. The aerolites being

infinitesimal, we do not consider the impact as between finite masses, but they constitute a medium through which the planet moves. Considering the medium as of uniform density, D, and the resistance varying with the square of the velocity, the case comes within one discussed by La Place, *Mécanique Céleste*, (8925, 8926).

The equations will become for this case,

$$da = -\frac{2KDa^2}{\mu}d\theta,$$

$$\frac{de}{e} = -\frac{KDa}{\mu}d\theta;$$

K being a constant depending upon the mass and form of the planet. If

$$\frac{KD}{\mu} = K',$$

$$\Delta a = \frac{2K'a_1^2\theta}{1 + 2K'a_1\theta},$$

$$e = \frac{e_1}{\sqrt{(1 + 2K'a_1\theta)}};$$

a_1 and e_1 being initial. The major axis and eccentricity both decrease as the vectorial angle increases, and the orbit becomes more nearly circular.

The plane of the orbit will not be changed; and, finally, the longitude of the perihelion will not be changed.—*Mécanique Céleste* (8916).

52. *Show that if two bodies revolve about a centre, acted upon by a force proportional to the distance from the centre, and independent of the mass of the attracted body, each will appear to the other to move in a plane, whatever may be their mutual attraction.*

Take the plane xy in the plane of the central force and of the two bodies at any instant, the origin at the central force and the axis of x passing through the body a, the coördinates of a being x' and 0, and of the other body b, x'' and y'', d their distance

apart ; M the central force at unit's distance, F the force of b on a, and F' that of a on b (according to the Newtonian law $F = F'$) ; X', X'', Y', Y'', the axial accelerations of a and b. Then

$$X' = -Mx' + F\frac{x''-x'}{d},$$

$$Y' = F\frac{y''}{d};$$

$$X'' = -Mx'' - F'\frac{x''-x'}{d},$$

$$Y'' = -My'' - F'\frac{y''}{d};$$

$$\therefore \frac{Y''-Y'}{X''-X'} = \frac{y''}{x''-x'},$$

which gives the direction of the relative accelerations, and which is parallel to the line ab. Hence, whatever be the directions of motion of the two bodies or their absolute velocities, their relative positions (a, b), their relative velocities, and their relative accelerations are parallel to a plane.

Solution by Quaternions.—Let ρ and ρ_1 be the vectors of the two bodies referred to the centre as origin. Let $\rho' = \dfrac{d\rho}{dt}$, $\rho_1' = \dfrac{d\rho_1}{dt}$, $\rho'' = \dfrac{d\rho'}{dt}$, $\rho_1'' = \dfrac{d\rho_1'}{dt}$. If M is the central force at the unit of distance, and N and N_1 the mutual attractions divided by the distance apart, we have

$$\rho'' = -M\rho + N(\rho_1 - \rho),$$

$$\rho_1'' = -M\rho_1 + N_1(\rho - \rho_1);$$

$$\therefore \rho_1'' - \rho'' = -M(\rho_1 - \rho) + (N_1 + N)(\rho - \rho').$$

The scalar part gives

$$S[(\rho-\rho')(\rho'-\rho_1')(\rho''-\rho_1'')] = (M+N+N_1)S(\rho-\rho_1)^2(\rho'-\rho'_1) = 0,$$

which proves that the apparent orbit is a plane.—(*Coördinate Geometry*, p. 278, eq. (3).) (*Math. Visitor*, July, 1880.)

53. *An elastic string, without weight and of given length, has one end fixed in a perfectly smooth horizontal plane, and the other to a point in the surface of a sphere, the string being unwound. The sphere is projected on the plane from the fixed point with a linear velocity v, and an angular velocity ω, winding the string on the circumference of a great circle ; required the elongation of the string when fully stretched, and the subsequent motion of the sphere.*

Let $r =$ the radius of the sphere, $a =$ the original length of the string, $\omega =$ the initial angular velocity of the body, $v =$ the initial velocity of the centre of the body, and $t_1 =$ the time of winding the slack. Then

$$vt_1 + r\omega t_1 = a;$$

$$\therefore t_1 = \frac{a}{v + r\omega},$$

and the initial stretched part will be

$$vt_1 = \frac{va}{v + r\omega} = l \text{ (say)}.$$

Immediately following this time the string will be stretched, and the tension at first diminishes both the linear and angular velocities. Take the origin at the remote end of l for the variable motion. Let $m =$ the mass of the body, $s =$ the space passed over by the centre during time t, $\theta =$ the angular distance passed by the initial radius in the same time, $k =$ the radius of gyration of the body, $e =$ the coefficient of elasticity of the string, A the cross section of the string, and λ the elongation produced by the tension T of the string. Then Mariotte's law gives

$$T = \frac{eA\lambda}{l - r\theta}. \tag{1}$$

Assume that l is so long compared to $r\theta$, that the latter can be neglected, and let $B = eA \div l$, then

$$T = B\lambda.$$

The conditions of the problem give

$$d\lambda = ds + rd\theta\,; \tag{2}$$

$$\therefore\ d^2\lambda = d^2s + rd^2\theta.$$

Also, for motion of the centre,

$$m\frac{d^2s}{dt^2} = -T = -B\lambda, \tag{3}$$

and for the rotary motion,

$$mk^2\frac{d^2\theta}{dt^2} = -Tr = -Br\lambda, \tag{4}$$

which two equations in the preceding give

$$\frac{d^2\lambda}{dt^2} = -\frac{B}{mk^2}(k^2 + r^2)\lambda = -D^2\lambda.$$

Integrating, observing that for $\lambda = 0$, $t = 0$, and $d\lambda \div dt = v + r\omega$, we have

$$\lambda = \frac{v + r\omega}{D}\sin Dt. \tag{5}$$

The elongation λ will be a maximum for $\sin Dt = 1$, or $t = \pi \div 2D$, for which

$$\lambda = \frac{v + r\omega}{D} = \frac{k\sqrt{ml}}{\sqrt{[eA(k^2 + r^2)]}}(v + r\omega).$$

The time of producing the maximum stretch of the string is independent of the initial motions. When the string returns to its original length λ will again be zero, and $\sin Dt = 0$, or $Dt = \pi$; $\therefore\ t = \dfrac{\pi}{D}$.

All the circumstances of the variable motion may be determined by integrating equations (3) and (4). Integrating after substituting from equation (5), observing that for $t = 0$, $\dfrac{ds}{dt} = v$, $s = 0$, $\dfrac{d\theta}{dt} = \omega$, and $\theta = 0$, we have, if we put F for $eA(v + r\omega) \div mlD^3$,

$$\frac{ds}{dt} = F[\cos Dt - 1] + v, \tag{6}$$

$$s = \frac{F}{D}[\sin Dt - Dt] + vt, \tag{7}$$

$$\frac{d\theta}{dt} = F\frac{r}{k^2}[\cos Dt - 1] + \omega, \tag{8}$$

$$\theta = \frac{F}{D}\frac{r}{k^2}[\sin Dt - Dt] + \omega t. \tag{9}$$

For the maximum of (5) $d\lambda \div dt = 0$, which in (2) gives

$$\frac{ds}{dt} = -r\frac{d\theta}{dt},$$

which combined with (6) and (8) gives $\cos Dt - 1 = -1$; $\therefore\ Dt = \frac{1}{2}\pi$ as before found, and serves as a check upon the work. The relation $ds = -rd\theta$ shows that the direction of one of the motions changes sign. At the point where the linear motion is reversed, $\frac{ds}{dt} = 0$, and for this we have

$$t_2 = \frac{1}{D}\cos^{-1}\left(1 - \frac{v}{F}\right);$$

and if the direction of rotation is reversed, $\frac{d\theta}{dt} = 0$, and (8) gives

$$t_3 = \frac{1}{D}\cos^{-1}\left(1 - \frac{\omega k^2}{Fr}\right);$$

from which it appears that if $v < k^2\omega \div r$ the motion of the centre will be reversed, but otherwise the angular motion will be reversed. The value of t_2 in the former case will be less than $\frac{\pi}{2D}$. Both motions will change at the instant of greatest elongation if $rv = k^2\omega$.

If the values of t_2 and t_3 are both less than $\pi \div D$, one motion will change sign before the instant of greatest elongation and the other after; otherwise only one will change sign. To find

the total variable movement, make $Dt = \pi$, and (7) and (9) give

$$s = (v - F)\frac{\pi}{D},$$

$$\theta = \left(\omega - F\frac{r}{k^2}\right)\frac{\pi}{D}.$$

If (6) reduces to zero when $Dt = \pi$, the body would rest at the moment the string regains its original length, and $F = \frac{1}{2}v$, but it would still have an angular velocity of $\omega + (rv \div k^2)$, as shown by (8). Similarly, if the rotary motion is destroyed at that instant, the linear velocity will be $v + (k^2\omega \div r)$, and will continue uniform. It may be shown that the kinetic energy of the moving body at the end of the variable motion is the same as at the beginning.

It has not been attempted to solve the general case represented by equation (1). It is evidently intricate.—(Problem and solution by the author in *The Analyst*, Jan., 1882.)

54. *Find the minimum eccentricity of an ellipse capable of resting in equilibrium on a perfectly rough inclined plane, inclination β.*

The centre of the ellipse will be vertically over the point of support, and since the plane is tangent to the ellipse, a vertical through the point of support and a parallel to the plane through the centre will make conjugate diameters. Hence the acute angle of the conjugate diameters is $90° - \beta$.

At the point bordering on motion, the potential energy is a maximum, and the major axis will bisect the acute angle of the conjugate diameters; hence the positive angles made by the conjugate diameters with the major axis will be $\theta = 45° - \frac{1}{2}\beta$, $\theta' = 135° + \frac{1}{2}\beta$.

The condition for conjugate diameters is

$$a^2 \sin\theta \sin\theta' + b^2 \cos\theta \cos\theta' = 0,$$

which, by substituting the preceding values, gives

$$a^2 \sin^2(45° - \tfrac{1}{2}\beta) - b^2 \cos^2(45° - \tfrac{1}{2}\beta) = 0;$$

which, reduced, gives

$$\sqrt{\left(\frac{a^2 - b^2}{a^2}\right)} = e = \sqrt{\left(\frac{2 \sin \beta}{1 + \sin \beta}\right)}.$$

(*Math. Visitor*, Jan., 1879.)

FLUIDS.

55. *A sphere* 4 *inches in diameter, specific gravity* 0.2, *is placed* 10 *feet under water. If left free to move, what will be its velocity at the surface of the water, and what will be the maximum height it will attain.*

Let $r = \frac{1}{6}$ = radius of the sphere, $\rho = \frac{1}{5}$ = its specific gravity, $h = 10$ feet, v the velocity acquired in ascending a distance x, V the velocity at the surface of the water, k = the resistance.

For the motion of the sphere,

$$\frac{vdv}{dx} = g\left(\frac{1}{\rho} - 1\right) - kv^2.$$

Putting $g\left(\frac{1}{\rho} - 1\right) = g'$,

$$dx = \frac{vdv}{g' - kv^2},$$

$$\int_0^h dx = \int_0^V \frac{vdv}{g - kv^2},$$

$$h = \frac{1}{2k} \log\left(\frac{g'}{g' - kV^2}\right);$$

whence,

$$V^2 = \frac{g'}{k}(1 - e^{-2hk}),$$

$$= \frac{g(1-\rho)}{k\rho} \text{ nearly.}$$

The required height is

$$\frac{V^2}{2g} = \frac{1-\rho}{2k\rho}(1 - e^{-2hk}),$$

$$= \frac{1-\rho}{2k\rho} \text{ nearly.}$$

The value of k reduced from Newton's Principia, Book 2, Prop. 38, is $\dfrac{3}{16r\rho}$, which gives

$$V = [\tfrac{16}{3} rg(1-\rho)]^{\frac{1}{2}} = \frac{4}{9}\sqrt{6g},$$

and

$$h = \frac{v^2}{2g} = 4\tfrac{4}{5} \text{ inches nearly.}$$

(*Math. Visitor*, Jan., 1879.)

56. *The first of two casks contains* a *gallons of wine, and the second* b *gallons of water ;* c *gallons were drawn from the second cask, and then* c *gallons were drawn from the first cask and poured into the second, and the deficiency in the first supplied by* c *gallons of water ;* c *gallons were then drawn from the first cask, and* c *gallons from the second, and poured into the first, and the deficiency in the second cask supplied by* c *gallons of wine. Required the quantity of wine in each cask after* n *such operations as that described above.*

Let u_n and v_n represent the wine in the first and second casks respectively at the end of the nth operation ; the quantities of wine in each cask at the successive stages of the $(n+1)$th operation are

$$u_n,\ v_n ;\ \left(1-\frac{c}{a}\right)u_n,\ \left(1-\frac{c}{b}\right)v_n ;\ \left(1-\frac{c}{a}\right)u_n,\ \left(1-\frac{c}{a}\right)v_n + \frac{c}{a}u_n ;$$

$$\left[\left(1-\frac{c}{a}\right)^2 + \frac{c^2}{ab}\right]u_n + \frac{c}{b}\left(1-\frac{c}{b}\right)v_n,\ \left(1-\frac{c}{b}\right)^2 v_n + \frac{c}{a}\left(1-\frac{c}{b}\right)u_n + c.$$

Whence

$$u_{n+1} = \left[\left(1-\frac{c}{a}\right)^2 + \frac{c^2}{ab}\right]u_n + \frac{c}{b}\left(1-\frac{c}{b}\right)v_n, \tag{1}$$

$$v_{n+1} = \left(1 - \frac{c}{b}\right)^2 v_n + \frac{c}{a}\left(1 - \frac{c}{b}\right)u_n + c. \tag{2}$$

Also

$$u_{n+2} = \left[\left(1 - \frac{c}{a}\right)^2 + \frac{c^2}{ab}\right]u_{n+1} + \frac{c}{b}\left(1 - \frac{c}{b}\right)v_{n+1}, \tag{3}$$

$$v_{n+2} = \left(1 - \frac{c}{b}\right)^2 v_{n+1} + \frac{c}{a}\left(1 - \frac{c}{b}\right)u_{n+1} + c. \tag{4}$$

Eliminating v_n from (1) and (2),

$$\left(1 - \frac{c}{b}\right)u_{n+1} - \frac{c}{b}\, v_{n+1} = \left(1 - \frac{c}{a}\right)^2\left(1 - \frac{c}{b}\right) u_n - \frac{c^2}{b}. \tag{5}$$

Eliminating v_{n+1} from (3) and (5),

$$u_{n+2} - \left[\left(1 - \frac{c}{a}\right)^2 + \frac{c^2}{ab} + \left(1 - \frac{c}{b}\right)^2\right]u_{n+1} + \left(1 - \frac{c}{a}\right)^2\left(1 - \frac{c}{b}\right)^2 u_n$$

$$= \frac{c^2}{b}\left(1 - \frac{c}{b}\right). \tag{6}$$

Let r_1, r_2, be the roots of the equation

$$y^2 - \left[\left(1 - \frac{c}{a}\right)^2 + \frac{c^2}{ab} + \left(1 - \frac{c}{b}\right)^2\right]y + \left(1 - \frac{c}{a}\right)^2\left(1 - \frac{c}{b}\right)^2 = 0.$$

The solution of (6) is, (Hymer's *Finite Differences*, pp. 54, 55),

$$u_n = C_1 r_1^n + C_2 r_2^n + \left\{\frac{\frac{c^2}{b}\left(1 - \frac{c}{b}\right)}{1 - \left(1 - \frac{c}{a}\right)^2 - \frac{c^2}{ab} - \left(1 - \frac{c}{b}\right)^2 + \left(1 - \frac{c}{a}\right)^2\left(1 - \frac{c}{b}\right)^2}\right\}.$$

Let the last expression $= S$. From (1), $u_0 = a$,

$$u_1 = \frac{(a - c)^2}{a} + \frac{c^2}{b}. \quad C_1 + C_2 = a - S, \; r_1 C_1 + r_2 C_2 = u_1 - S;$$

whence

$$C_1 = \frac{u_1 - S - r_2(a - S)}{r_1 - r_2},$$

$$C_2 = \frac{r_1(a - S) - u_1 + S}{r_1 - r_2},$$

and

$$u_n = \left(\frac{u_1 - S - r_2(a - S)}{r_1 - r_2}\right) r_1^n + \left(\frac{r_1(a - S) - u_1 + S}{r_1 - r_2}\right) r_2^n + S. \qquad (7)$$

Eliminating u_n from (1) and (2), and u_{n+1} from this and (4),

$$v_{n+2} - \left[\left(1 - \frac{c}{a}\right)^2 + \frac{c^2}{ab} + \left(1 - \frac{c}{b}\right)^2\right] v_{n+1} + \left(1 - \frac{c}{a}\right)^2 \left(1 - \frac{c}{b}\right)^2 v_n$$

$$= c\left[1 - \left(1 - \frac{c}{a}\right)^2 - \frac{c^2}{ab}\right],$$

whence

$$v_n = C_3 r_1^n + C_4 r_2^n + S_1.$$

From (2),

$$v_0 = 0, \quad v_1 = c\left(2 - \frac{c}{b}\right);$$

therefore

$$v_n = \left(\frac{v_1 + (r_2 - 1)S_1}{r_1 - r_2}\right) r_1^n + \left(\frac{S_1(1 - r_1) - v_1}{r_1 - r_2}\right) r_2^n + S_1.$$

(*Math. Visitor*, Jan., 1880.)

57. *A servant draws off a gallon a day for* 20 *days, from a cask holding* 10 *gallons of wine, adding each time a gallon of water to the cask. He then draws off* 20 *gallons more, adding as taken, a gallon of wine to the cask for each gallon drawn. How much water remains in the cask?*

Put 10 gallons $= a$. 1 gallon $= b$. $20 = t$, and let $u_x =$ the number of gallons of water in the cask at the end of the xth operation. Then we have

$$\left(\frac{a - b}{a}\right) u_x + b = u_{x+1}, \qquad (1)$$

an equation in Finite Differences. Integrating (1),

$$u_x = C\left(\frac{a-b}{a}\right)^x + a. \tag{2}$$

When

$$x = 0, \quad u_x = 0\,; \quad \therefore\ C = -a.$$

$$\therefore\ u_x = a\left(\frac{a^x - (a-b)^x}{a^x}\right) = c,$$

by the first condition.

Let v_t = the quantity of water in the cask at the end of the tth operation under the second condition; then,

$$\frac{a-b}{a}v_t = v_{t+1}.$$

Integrating,

$$v_t = C\left(\frac{a-b}{a}\right)^t.$$

When

$$t = 0, \quad v_0 = c\,;$$

$$\therefore\ C = c_1, \quad v_t = c\left(\frac{a-b}{a}\right)^t,$$

$$= a\left(\frac{a-b}{a}\right)^t\left(\frac{a^x - (a-b)^x}{a^x}\right),$$

When

$$x = t = 20, \quad a = 10, \quad b = 1,$$

$$v_t = 10\,(.9^{20} - .9^{40}).$$

(*The Wittenberger*, Jan., 1880.)

58. *A piston, weight w, is dropped into the end of a vertical cylinder filled with air, length l; how far will the piston descend, assuming no friction nor escape of air, nor heat from the compressed air?*

There being no escape of heat, the law of pressure will be expressed by

$$pv^k = \text{constant} = p'v'^k, \tag{1}$$

where p' = the initial pressure of the atmosphere = 15 lbs. nearly ; v' = volume of the cylinder = al, if a is the area of the base ; p = the pressure within the cylinder when the weight has descended a distance x ; $a(l-x)$ = the volume when the pressure is p ; $k = 1.408$; then, from (1),

$$p = \frac{p'l^k}{(l-x)^k},$$

and

$$\frac{wd^2x}{gdt^2} = w + p'a - \frac{ap'l^k}{(l-x)^k}.$$

Integrating, observing that for $x = 0$, $v = 0$,

$$v^2 = \left(1 + \frac{p'a}{w}\right)2gx + \frac{2ap'g}{w(k-1)}\left(\frac{l(l-x)^{k-1} - l^k}{(l-x)^{k-1}}\right).$$

At the end of the downward movement $v = 0$; therefore

$$x(w + pa)(l-x)^{k-1} + \frac{ap'l}{k-1}(l-x)^{k-1} = \frac{ap'l^{k+1}}{k-1},$$

from which x may be found by trial after numerical values have been substituted for the known quantities.

59. *If each of* n *vessels closely connected in circuit contains a different liquid, each* q *gallons, and the liquids circulate by flowing uniformly in one direction at the rate of* a *gallons per minute, mixing uniformly, how much of each liquid will there be in any one of the vessels at the end of the time* t.

Let the vessels be numbered in the natural order 1, 2, 3 . . . n ; let x denote any particular liquid and x_1, x_2, x_3, . . . x_n, the quantity of it in vessels 1, 2, etc, respectively at time t. adt = the

amount flowing out (and in) each instant, of which $\frac{x_1}{q}adt$, $\frac{x_2}{q}adt \ldots \frac{x_n}{q}adt$, will be of the liquid denoted by x. In vessel 1, $\frac{x_1}{q}adt$ flows out, and $\frac{x_n}{q}adt$ flows in from the nth vessel, and the difference will be an element of the decrease of the x liquid;

$$\therefore\ dx_1 = \frac{x_1}{q}adt - \frac{x_n}{q}adt. \qquad (a)$$

Letting $\frac{q}{a} = r$, and dividing by dt, we have

$$\left.\begin{array}{l} \dfrac{rdx_1}{dt} = x_1 - x_n, \\ \text{or} \\ \dfrac{rdx_1}{dt} - x_1 = -x_n. \\ \text{Similarly for the others,} \\ \dfrac{rdx_2}{dt} - x_2 = -x_1 \\ \dfrac{rdx_3}{dt} - x_3 = -x_2 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \dfrac{rdx_n}{dt} - x_n = -x_{n-1} \end{array}\right\}. \qquad (b)$$

Differentiating $n - 1$ times, we have

$$\left.\begin{array}{l} \dfrac{rd^nx_1}{dt^{n-}} - \dfrac{d^{n-1}x_1}{dt^{n-1}} = -\dfrac{d^{n-1}x_n}{dt^{n-1}} \\ \dfrac{rd^{n-1}x_1}{dt^{n-1}} - \dfrac{d^{n-2}x_1}{dt^{n-2}} = -\dfrac{d^{n-2}x_n}{dt^{n-2}} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ \dfrac{rdx_1}{dt} - x_1 = -x_n \end{array}\right\}. \qquad (c)$$

Similarly,

$$\frac{rd^n x_2}{dt^n} - \frac{d^{n-1}x_2}{dt^{n-1}} = -\frac{d^{n-1}x_1}{dt^{n-1}},$$

$$\frac{rd^{n-1}x_2}{dt^{n-1}} - \frac{d^{n-2}x_2}{dt^{n-2}} = -\frac{d^{n-2}x_1}{dt^{n-2}},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{rdx_2}{dt} - x_2 = -x_1;$$

and finally,

$$\frac{rd^n x_{n-1}}{dt^n} - \frac{d^{n-1}x_{n-1}}{dt^{n-1}} = -\frac{d^{n-1}x_{n-2}}{dt^{n-1}},$$

$$\frac{rd^{n-1}x_{n-1}}{dt^{n-1}} - \frac{d^{n-2}x_{n-1}}{dt^{n-2}} = -\frac{d^{n-2}x_{n-2}}{dt^{n-2}},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{rdx_{n-1}}{dt} - x_{n-1} = -x_{n-2}.$$

Multiplying the successive equations in (c) by the successive terms of the development of the binomial $(r-1)^{n-1}$, and adding results, we have

$$\frac{r^n d^n x_1}{dt^n} - \frac{nr^{n-1}d^{n-1}x_1}{dt^{n-1}} + \frac{n(n-1)}{\lfloor 2} \frac{r^{n-2}d^{n-2}x_1}{dt^{n-2}} \quad \cdot \cdot \cdot \cdot \cdot \pm \frac{nrdx_1}{dt} \mp x_1$$

$$= -\frac{r^{n-1}d^{n-1}x_n}{dt^{n-1}} + (n-1)\frac{r^{n-2}d^{n-2}x_n}{dt^{n-2}} - \frac{(n-1)(n-2)}{\lfloor 2} \frac{r^{n-3}d^{n-3}x_n}{dt^{n-3}}$$

$$\pm (n-1)\frac{rdx_n}{dt} \mp x_n.$$

The second member of this equation is one degree less than the first, and since this is true for all values of n, it will be

true if we reduce the exponent and subscript still further by 1. This process ultimately gives for the second member,

$$\pm \frac{rdx_2}{dt} \mp x_2 = \mp x_1 ;$$

hence we have

$$\frac{r^n d^n x_1}{dt^n} - \frac{nr^{n-1}d^{n-1}x_1}{dt^{n-1}} + \frac{n(n-1)}{\lfloor 2} \frac{r^{n-2}d^{n-2}x_1}{dt^{n-2}} \cdots$$

$$\mp \frac{n(n-1)}{\lfloor 2} \frac{r^2 d^2 x_1}{dt^2} \pm \frac{nrdx_1}{dt} = 0.$$

Adding ∓ 1 to both members, the characteristic equation becomes (Price's *Infinitesimal Calculus*, Vol. II., p. 634),

$$r^n \beta^n - nr^{n-1}\beta^{n-1} \ldots\ldots \pm nr\beta \mp 1 = \mp 1,$$

or

$$(r\beta - 1)^n = \mp 1 ;$$

$$\therefore \beta = \frac{1 + \sqrt[n]{\mp 1}}{r},$$

which may be written

$$\beta = \frac{1 + \sqrt[n]{(-1)^n}}{r}, \tag{d}$$

where the exponent n, of -1, is simply for the purpose of determining the sign of this term. If n be even $(-1)^n$ will be $+1$, and there will be two real roots $+1$ and -1 ; if n be odd there will be one real root of $\sqrt[n]{-1} = -1$. The other roots will be imaginary and in pairs. The number of values of β will equal the degree of the equation, one value being zero, and letting these values be $b_1, b_2, b_3 \ldots b_{n-1}$, the integral becomes (Price's *Infinitesimal Calculus*, Vol. II., eq. (107), p. 637),

$$x_1 = C_1 e^{b_1 t} + C_2 e^{b_2 t} + \ldots\ldots + C_n e^{b_n t}, \tag{e}$$

and similar expressions for x_2, x_3, etc.

To find the constants of integration we have for $t = 0$, $x_1 = q$, $\frac{dx_1}{dt} = a$, from (a) since, initially, $x_n = 0$,

$$\left.\frac{d^2x_1}{dt^2}\right]_{t=0} = \left.\left(\frac{a}{q}\frac{dx_1}{dt} - \frac{a}{q}\frac{dx_n}{dt}\right)\right]_{t=0} = \frac{a^2}{q} - 0 = \frac{a^2}{q};$$

$$\left.\frac{d^3x_1}{dt^3}\right]_{t=0} = \frac{a^3}{q^2}, \text{ etc.} \qquad (f)$$

$$\therefore\ q = C_1 + C_2 + C_3 \quad . \quad . \quad . \quad . \quad C_n,$$

$$a = b_1C_1 + b_2C_2 \quad . \quad . \quad . \quad . \quad b_nC_n,$$

$$\frac{a^2}{q} = b_1^2C_1 + b_2^2C_2 \quad . \quad . \quad . \quad . \quad b_n^2C_n,$$

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

and finally

$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$

$$\frac{a^n}{q^{n-1}} = b_1^nC_1 + b_2^n \quad . \quad . \quad . \quad . \quad . \quad b_n^nC_n,$$

from which the constants of integration can be found

As a special case let $n = 3$, then from (d),

$$\beta = 0, \quad \frac{a}{2q}(3 - \sqrt{-3}), \quad \text{and } \frac{a}{2q}(3 + \sqrt{-3}).$$

We then have from (e),

$$x_1 = C_1 + C_2e^{\frac{a}{2q}(3-\sqrt{-3})t} + C_3e^{\frac{a}{2q}(3+\sqrt{-3})t},$$

and similarly,

$$x_2 = A_1 + A_2e^{\frac{a}{2q}(3-\sqrt{-3})t} + A_3e^{\frac{a}{2q}(3+\sqrt{-3})t} \qquad (g)$$

$$x_3 = B_1 + B_2e^{\frac{a}{2q}(3-\sqrt{-3})t} + B_3e^{\frac{a}{2q}(3+\sqrt{-3})t}$$

The sum of these, or $x_1 + x_2 + x_3 = q$ at any time t.

Also by the process shown in (f), we have for $t = 0$, $x_1 = q$, $\frac{dx_1}{dt} = a$, $\frac{d^2x_1}{dt^2} = \frac{a^2}{q}$; $x_2 = 0$, $\frac{dx_2}{dt} = -a$, $\frac{d^2x_2}{dt^2} = -\frac{2a^2}{q}$; $x_3 = 0$, $\frac{dx_3}{dt} = 0$, $\frac{d^2x_3}{dt^2} = \frac{a^2}{q}$.

Hence, to find the constants in equation (g), we have. for x_1,

$$q = C_1 + C_2 + C_3,$$

$$a = \frac{a}{2q}(3 - \sqrt{-3})C_2 + \frac{a}{2q}(3 + \sqrt{-3})C_3,$$

$$\frac{a^2}{q} = \frac{a^2}{4q^2}(3 - \sqrt{-3})^2 C_2 + \frac{a^2}{4q^2}(3 + \sqrt{-3})^2 C_3;$$

for x_2,

$$0 = A_1 + A_2 + A_3,$$

$$-a = \frac{a}{2q}(3 - \sqrt{-3})A_2 + \frac{a}{2q}(3 + \sqrt{-3})A_3,$$

$$-\frac{2a^2}{q} = \frac{a^2}{4q^2}(3 - \sqrt{-3})^2 A_2 + \frac{a^2}{4q^2}(3 + \sqrt{-3})^2 A_3;$$

and for x_3,

$$0 = B_1 + B_2 + B_3,$$

$$0 = \frac{a}{2q}(3 - \sqrt{-3})B_2 + \frac{a}{2q}(3 + \sqrt{-3})B_3,$$

$$\frac{a^2}{q} = \frac{a^2}{4q^2}(3 - \sqrt{-3})^2 B_2 + \frac{a^2}{4q^2}(3 + \sqrt{-3})^2 B_3.$$

These give

$$C_1 = C_2 = C_3 = \tfrac{1}{3}q.$$

$$A_1 = \tfrac{1}{3}q, \quad A_2 = -\tfrac{1}{6}q(1 + \sqrt{-3}), \quad A_3 = -\tfrac{1}{6}q(1 - \sqrt{-3}).$$

$$B_1 = \tfrac{1}{3}q, \quad B_2 = -\tfrac{1}{6}q(1 - \sqrt{-3}), \quad B_3 = -\tfrac{1}{6}q(1 + \sqrt{-3}).$$

These in (g) give

$$x_1 = \tfrac{1}{3}q\left[1 + e^{\frac{3at}{2q}}\left(e^{\frac{at}{2q}\sqrt{-3}} + e^{-\frac{at}{2q}\sqrt{-3}}\right)\right].$$

$$= \tfrac{1}{3}q\left(1 + 2e^{\frac{3at}{2q}} \cos\frac{\sqrt{3}at}{2q}\right);$$

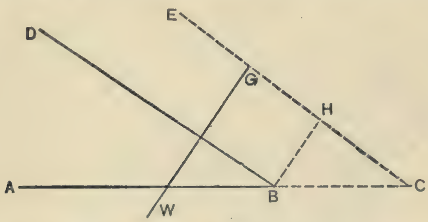
$$x_2 = \tfrac{1}{6}q\left[2 - 2e^{\frac{3at}{2q}}\left(\cos\frac{\sqrt{3}at}{2q} + \sqrt{3}\sin\frac{\sqrt{3}at}{2q}\right)\right];$$

$$x_3 = \tfrac{1}{6}q\left[2 - 2e^{\frac{3at}{2q}}\left(\cos\frac{\sqrt{3}at}{2q} - \sqrt{3}\sin\frac{\sqrt{3}at}{2q}\right)\right].$$

In a similar manner, if y is the liquid in the second vessel, the quantity of it in vessel 3 at the end of time t will be $y_1 = x_1$ above; and similarly for the others.

60. *To find the velocity of an ice boat.*

Let AB represent the track of the boat, BD the position of the sail, $\theta = DBA$ and WG the direction of the wind, which we will assume to be normal to the sail. When the boat advances to C, the position of the sail will be CE. If V be the velocity of the boat, proportional to BC, and v the velocity of the wind relatively to the earth, then will



$$v - V\sin\theta$$

be the velocity relatively to the sail, since the wind passing any point as B must travel a distance BH before coming in contact with the sail. The pressure of the wind is assumed to vary as the square of the velocity relatively to the surface pressed, and if M be the mass of the boat and sail, B a constant depending

upon the size of the sail and the unit of velocity, and neglecting all resistances, we have

$$M\frac{d^2s}{dt^2} = B(v - V\sin\theta)^2;$$

$$\therefore \frac{d\left(\frac{ds}{dt}\right)}{\left(\frac{v}{\sin\theta} - \frac{ds}{dt}\right)^2} = A\,dt,$$

where $A = \frac{B\sin^2\theta}{M}$. Integrating, making $V = 0$ for $t = 0$,

$$\frac{ds}{dt} = V = \frac{\frac{v}{\sin\theta}}{\frac{\sin\theta}{Avt} + 1};$$

from which it will be seen that V increases as t increases indefinitely, the limit of V being

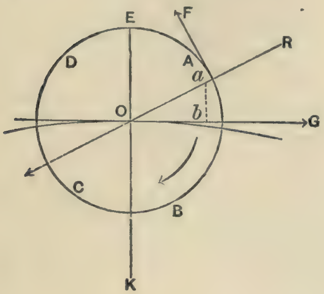
$$V = \frac{v}{\sin\theta};$$

and this increases as θ decreases, from which it appears that, according to the above hypothesis, the boat might attain an immense velocity for a small velocity of the wind. The smaller the angle of the sail with the track of the boat, the greater the ultimate velocity of the latter, there being no resistances.

But the resistances may be considerable. The coefficient of friction on the ice may possibly be as low as 0.04. Should the air move away from behind the sail with the velocity of the wind, or even with the velocity $V\sin\theta$, no resistance would be offered by the air; but such will not be the case. The mast, sail, and other parts will be opposed by the resistance of the air as they move through it, but it is difficult to determine the exact amount. General Tower, in Van Nostrand's *Engineering Magazine*, January, 1880, pp. 83, 84, according to an example of probable conditions, concluded that the maximum velocity of an ice boat might exceed twice that of the wind.

61. *Determine the path of a rotating body projected into a resisting medium.*

A general solution of this problem has not been obtained, but certain qualitative results may be determined without finding the quantitative. Let the body be a sphere rotating about a vertical axis, the centre moving in a horizontal plane. Let ω be the angular velocity, and v the velocity of the centre; then will OK, the distance of the spontaneous axis K from the centre be, equation (199),



$$OK = \frac{v}{\omega}.$$

The combined motion of rotation and translation of the body at any instant being considered the same as that of the entire body rotating an infinitesimal amount about the axis K, the quadrants A and D will move with equal velocities exceeding those of B and C. The resistance of the medium to a body moving normally against it will vary as some power of the velocity of the body, and in this case may be considered as *proportional* to the same. The quadrant A moves against the medium with a greater velocity than B, hence the pressure on that quadrant will be greater, while the velocity of the quadrant D moving away from the medium, exceeds that of C. Therefore the pressure of the medium will be greatest on quadrant A, next B, then C, and least on D. The resultant of these pressures will not be zero, and generally not parallel to OG. Let R be the resultant, the component of which, parallel to OG, will be the pressure directly opposing the motion, and ab, normal to OG, the pressure which will deflect it from its initial direction. Neglecting friction, the path will be a curve convex towards the quadrant of greatest pressure, and will be more nearly a right line as K is more remote, or the more v exceeds ω. Still, neglecting friction, the rotary motion will be constant, while the velocity of the centre will be diminished; hence the curvature of the path will increase with the distance traveled

The friction between the medium and body tends directly to

diminish the rotation, but if the sum of the components of the frictional resistances resolved in reference to two rectangular planes be not zero, there will be a resultant. The friction will be greatest where the pressure is greatest, as at A, and act tangentially to the surface. Let F be the resultant; it will be equal to a couple, and an equal parallel force at the centre, the former of which reduces the rotation; and of the latter, that component which is parallel to OG directly opposes the motion, and that which is normal to OG tends to deflect the path in the direction OE, opposite to that produced by pressure only. If the body be comparatively smooth and the medium rare, the friction will be only a fractional part of the pressure, and the *resultant friction* will be only a very small part of the entire friction, in which case the direction of the curvature of the path will be determined by the resultant *pressure*, but the amount of curvature will be diminished by the friction. This case is illustrated by a rotating sphere projected into air.

But if the body be rough and the medium dense, frictional resistance might exceed the pressure, in which case the direction of curvature would be determined by the resultant friction, the amount being modified by pressure. This could be illustrated by a wheel with flat vanes rotating about its axis, placed vertical and pushed along in water. Or still more strikingly, if a rough cylinder rotating about a vertical axis be pushed into a bank of earth, the tendency to a lateral motion might be almost entirely dependent upon the friction.

APPENDIX II.

THE POTENTIAL.

THE POTENTIAL.

THE term "potential function," or simply *the potential*, as used by Gauss and subsequent writers, is applied to a certain expression appearing in certain investigations involving forces depending upon some function of the distance between the bodies. It is of value in the higher investigations of the theory of attraction, hydromechanics, electricity, magnetism, and heat.

Before defining it definitely, take an example. Let m and m' be the masses respectively of two particles, the place of m being x, y, z, of m', x', y' z', the distance between them r, and $f(r)$ the law of the mutual attraction or repulsion. Then will the stress between them be

$$P = \mp mm'f(r), \tag{a}$$

$-$ being attractive and $+$ repulsive. The axial components will be

$$P \cos \alpha = X = mm'f(r)\frac{x' - x}{r}, \quad Y = mmf(r)\frac{y' - y}{r},$$

$$Z = mm'f(r)\frac{z' - z}{r}; \tag{b}$$

and for the distance between them,

$$r^2 = (x' - x)^2 + (y' - y)^2 + (z' - z)^2; \tag{c}$$

from which we have for the partial differential coefficients, x, y, z, being considered fixed,

$$rdr = (x' - x)dx', \quad rdr = (y' - y)dy', \quad rdr = (z' - z)dz. \tag{d}$$

Let $f(r)$ be considered as the differential coefficient of some other function of r, so that

$$\frac{dF(r)}{dr} = f(r); \tag{e}$$

then will equations (d) and (e) reduce (b) to

$$X = m'm \frac{dF(r)}{dx'}, \quad Y = m'm \frac{dF(r)}{dy'}, \quad Z = m'm \frac{dF(r)}{dz'}.$$

If m_2, m_3, etc., be the masses of other particles distant r_2, r_3 etc., from m', then we have, m being *typical* of m_2, m_3, etc.,

$$X = m' \frac{d\Sigma mF(r)}{dx'}, \quad Y = m' \frac{d\Sigma mF(r)}{dy'}, \quad Z = m' \frac{d\Sigma mF(r)}{dz'}. \qquad (f)$$

Let

$$\Sigma mF(r) = V, \qquad (g)$$

then we have

$$X = m' \left(\frac{dV}{dx'}\right), \quad Y = m' \left(\frac{dV}{dy'}\right), \quad Z = m' \left(\frac{dV}{dz'}\right); \qquad (h)$$

the parentheses indicating partial differential coefficients. The function V is called *the potential*, and may be defined by interpreting equation (g). When found, the axial components of the stress appear as partial differential coefficients of V regarded as a function of the coördinates of the particle. It is rarely used in this general form, but is confined to the cases where the law of the force is that of the inverse squares—the law most common in nature. Let the system of particles m be continuous, forming a solid, and the force attractive, then we have

$$f(r) = -\frac{1}{r^2}; \quad \therefore \ F(r) = \frac{1}{r};$$

and if δ be the density at the place x, y, z, then

$$dm = \delta dxdydz;$$

$$\therefore \ V = \iiint \frac{\delta dxdydz}{r}; \qquad (i)$$

that is, THE POTENTIAL *is sum of the quotients of all the elementary masses divided by their distance from the attracted particle.*

Again, if the mass of the attracted particle be unity, and that of the attracting particle be m, then at any distance r we have the stress,

$$P = -\frac{1 \cdot m}{r^2};$$

and hence an element of the work done upon the unit mass at that point in being moved over the space dr will be

$$dw = -\frac{m}{r^2}\,dr;$$

$$\therefore\ w = -\int_r^\infty \frac{m}{r^2}\,dr = \frac{m}{r}; \qquad (j)$$

and similarly for any number of particles forming a continuous body;

$$\therefore\ W = \Sigma w = \iiint \frac{\delta dxdydz}{r}; \qquad (k)$$

which is the same form as that above, hence, also,

THE POTENTIAL *is the energy acquired by a unit mass in falling from infinity under the attraction of a given body to a distance r.*

Again, m may represent any quantity of action, either attractive or repulsive, as in magnetism or electricity, in which the law of action is that of the inverse squares and product of their quantities.

From equation (c) we have the partial derivative

$$(dx') = \frac{rdr}{x' - x};$$

and from (i), considering V as a function of r,

$$(dV) = \iiint \frac{-\delta dxdydzdr}{r^2};$$

$$\therefore \left(\frac{dV}{dx'}\right) = -\iiint \frac{\delta(x' - x)dxdydz}{r^3};$$

similarly,

$$\left(\frac{d^2 V}{dx'^2}\right) = \iiint \left[\frac{3(x'-x)^2}{r^5} - \frac{1}{r^3}\right] dxdydz,$$

$$\left(\frac{d^2 V}{dy'^2}\right) = \iiint \left[\frac{3(y'-y)^2}{r^5} - \frac{1}{r^3}\right] dxdydz,$$

$$\left(\frac{d^2 V}{dz'^2}\right) = \iiint \left[\frac{3(z'-z)^2}{r^5} - \frac{1}{r^3}\right] dxdydz;$$

and adding

$$\left(\frac{d^2 V}{dx'^2}\right) + \left(\frac{d^2 V}{dy'^2}\right) + \left(\frac{d^2 V}{dz'^2}\right) = 0; \qquad (l)$$

which theorem was discovered by La Place. It is not general, however, for it is found to fail when the particle is one of the particles of the attracting mass; but it is correct when the particle attracted is external to the attracting body.

EXAMPLES.

1. *To find the potential of a slender uniform rod, length a, density δ, and section s, upon an external particle m'.*

Take the origin at one end of the rod, x, along the rod, and x', y', the position of the particle m'. Then

$$V = \int_0^a \frac{\delta s dx}{\sqrt{(x'-x)^2 + y'^2}} = \delta s \log \frac{a - x' + (y'^2 + (x'-a)^2)^{\frac{1}{2}}}{-x' + (x'^2 + x^2)^2}. \quad (m)$$

Hence the attractive forces parallel and normal to the rod will be respectively,

$$X = -m'\left(\frac{dV}{dx'}\right) = m'\delta s\left[\frac{1}{\sqrt{y'^2 + (a-x')^2}} - \frac{1}{\sqrt{y'^2 + x'^2}}\right].$$

$$Y = -m'\left(\frac{dV}{dy'}\right) = \frac{m'\delta s}{y'}\left[\frac{a-x'}{\sqrt{y'^2 + (a-x')^2}} - \frac{x'}{\sqrt{y'^2 + x'^2}}\right].$$

2. *To find the potential of a thin, homogeneous, spherical shell upon an external particle.*

Let $a =$ radius, $\delta =$ density, $da =$ thickness, $\rho =$ distance of particle from the centre of the shell. Using polar coördinates, origin at the centre of the shell, $(0, 0, \rho)$ the place of m', $\theta =$ polar distance of element of m from where ρ pierces the shell, $\varphi =$ longitude, initial at any point, then

$$dm = \delta da \cdot a \sin \theta d\varphi \cdot a d\theta,$$

$$r^2 = a^2 - 2a\rho \cos \theta + \rho^2 ;$$

$$\therefore \; V = \delta da \cdot a \int_0^{\pi} \int_0^{2\pi} \frac{\sin \theta d\theta d\varphi}{(a^2 - 2a\rho \cos \theta + \rho^2)^{\frac{1}{2}}},$$

$$= \frac{2\pi\delta a da}{\rho} \left(a^2 - 2a\rho \cos \theta + \rho^2\right)^{\frac{1}{2}}\Big]_0^{\pi},$$

$$= \frac{2\pi\delta a da}{\rho} \Big(a + \rho - (a - \rho)\Big), \qquad (n)$$

or,

$$= \frac{2\pi\delta a da}{\rho} \Big(a + \rho - (\rho - a)\Big). \qquad (o)$$

For an external particle $\rho > a$, hence the last equation is the correct form for this case, and the former gives the potential for a particle internal to the shell. For the latter,

$$V = \frac{4\pi\delta a^2 da}{\rho} = \frac{m}{\rho};$$

and for a concentric shell of finite thickness,

$$V = \frac{4\pi\delta}{\rho} \int_{a_2}^{a_1} a^2 da = \frac{4}{3} \frac{\pi\delta}{\rho} (a_1^3 - a_2^3) = \frac{M}{\rho};$$

$$\therefore \left(\frac{dV}{d\rho}\right) = -\frac{M}{\rho^2},$$

which is the force along the line ρ ; hence,

The attraction of a spherical, homogeneous shell upon an

exterior particle is the same as if the mass of the shell were condensed into a particle at the centre.

Equation (n) becomes

$$V = 4\pi\delta a da\,;$$

$$\therefore\ \left(\frac{dV}{d\rho}\right) = 0,$$

hence, ***The attraction of a spherical homogeneous shell upon a particle within it is zero.***